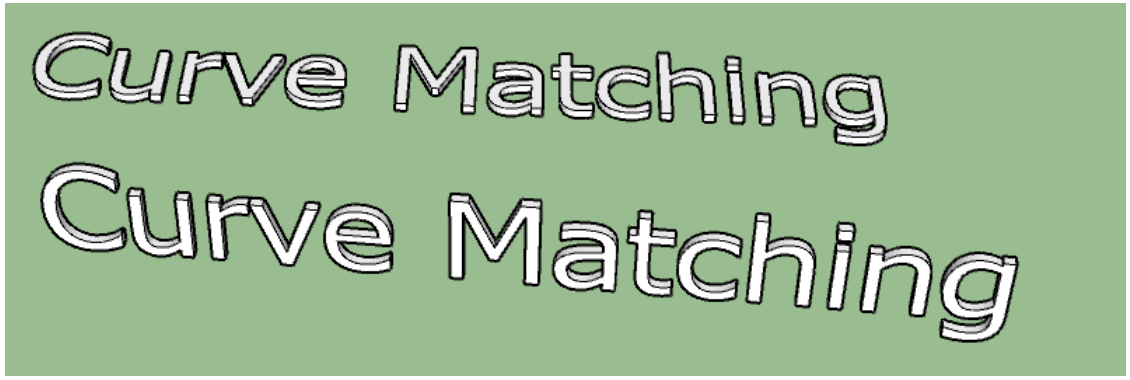

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Max-Planck-Institut für Mathematik in den Naturwissenschaften

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workshop on

Geometry of Curves in Time Series and Shape Analysis

Collaborators

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Outline

- The group equivalence and the projection problems for curves and the relationship between them.
- The equivalence problem for smooth parametrized curves under the group of rotations and translations ($SE(2)$-group).
  - The Frenét frame and the curvature.
  - The solution based on differential signature.
- The differential signature in the case of other groups.
- The differential signature of algebraic curves.
- The integral signatures and their numerical approximations.
- The projection problem revisited: relationship between invariants of an object and its image.

The group equivalence and the projection problems

and the relationship between them.

The group-equivalence problem for planar curves

G - a group acting on the affine or projective plane ($\mathbb{R}^2$, $\mathbb{PR}^2$, $\mathbb{C}^2$ or $\mathbb{PC}^2$).

$$\boxed{G \curvearrowright \text{plane} \Rightarrow G \curvearrowright \{\text{ planar curves }\}.}$$

Definition: Two curves X_1 and X_2 are G-equivalent (or G-congruent)

$$X_1 \underset{G}{\cong} X_2$$

if

$$\exists g \in G : X_1 = g \cdot X_2.$$

The G-equivalence problem:

- Given X_1, X_2 and G decide whether or not $X_1 \underset{G}{\cong} X_2$.

- Describe equivalence classes of curves.

Examples of groups and their actions: $(x, y) \mapsto (\bar{x}, \bar{y})$

$\boxed{E(2)\text{=Euclidean group}}$ acts by rotations, translations, reflections:

$$\boxed{\bar{x} = \cos(\phi)x - \sin(\phi)y + a, \qquad \bar{y} = \epsilon(\sin(\phi)x + \cos(\phi)y) + b}$$

$a, b, \phi \in \mathbb{R}$, $\epsilon = \pm 1$

If $\epsilon = 1$, then $SE(2)$= special Euclidean group

$\boxed{A(2)\text{=Affine group}}$ acts by invertible linear transformations and translations:

$$\boxed{\bar{x} = \alpha\, x + \beta\, y + a, \qquad \bar{y} = \gamma\, x + \delta\, y + b}$$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\delta - \beta\gamma \neq 0$

If $\alpha\delta - \beta\gamma = 1$, then $SA(2)$= special affine (or equi-affine) group

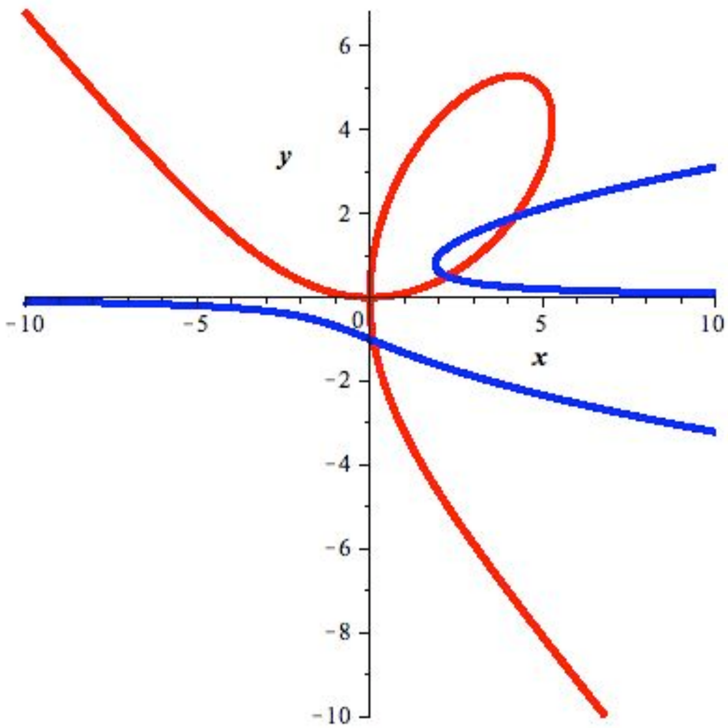
$\boxed{PGL(3)\text{=Projective group}= GL(3)\backslash\{\lambda I\}}$ acts by linear fractional transformations:

$$\boxed{\bar{x} = \frac{\alpha\, x+\beta\, y+a}{\nu\, x+\mu\, y+c}, \qquad \bar{y} = \frac{\gamma\, x+\delta\, y+b}{\nu\, x+\mu\, y+c}}, \qquad \det\begin{pmatrix} \alpha & \beta & a \\ \gamma & \delta & b \\ \nu & \mu & c \end{pmatrix} \neq 0$$

Problem: [*M. Berger, Geometry II, 1987)]* Locate four points on each of the following photograph, transfer them to a blank sheet of paper, and verify that the two sets of points cannot be mapped one to another by an affine transformation. They can be mapped to each other by a $\boxed{\text{projective transformation on the plane}}$.

Example of equivalence and non-equivalence:

$X_1 = \{(x,y) \,|\, x^3 + y^3 - 10xy = 0\}$ and $X_2 = \{(x,y) \,|\, y^3 - xy + 1 = 0\}$



$$X_1 \underset{E(2)}{\not\cong} X_2$$

and

$$X_1 \underset{A(2)}{\not\cong} X_2$$

but

$$X_1 \underset{PGL(3)}{\cong} X_2, \text{ with } X_2 = g \cdot X_1,$$

where

$$(\bar{x}, \bar{y}) = g \cdot (x,y) = \left(\frac{10}{y}, \frac{x}{y}\right)$$

Projections:

$P: \mathbb{P}^3 \to \mathbb{P}^2$

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$\mathrm{rank} P = 3$

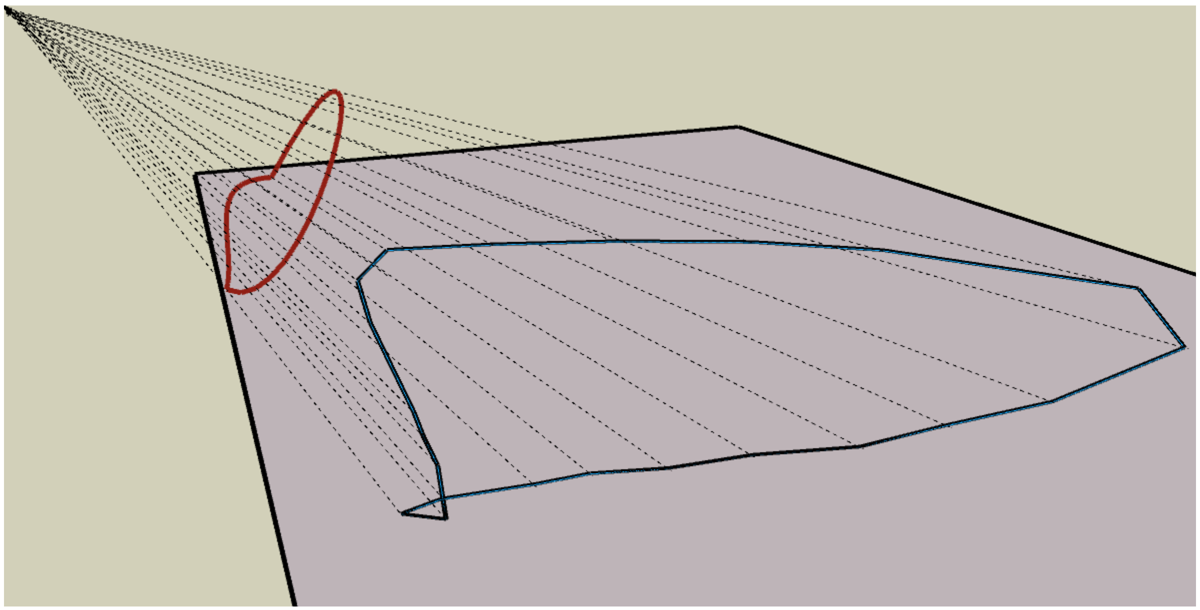
12 parameters p_{ij}, equivalent up to scaling by a nonzero constant $p_{ij} \to \lambda p_{ij}$.

The center is the kernel of P.

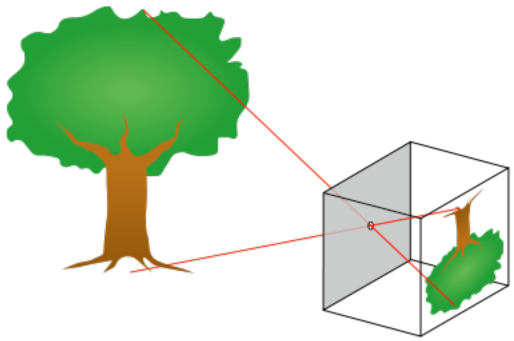
Projection (or object-image correspondence) problem for curves

Given a curve $Z \subset \mathbb{P}^3$ and a plane curve $X \subset \mathbb{P}^2$,

decide whether there exists $P\colon Z \to X$, such that $X = PZ$



Finite cameras, or central projection (the left 3×3 submatrix of P is non-singular):
$P\colon (z_1, z_2, z_3) \mapsto (x, y)$.



[Image from the Wikipedia]

$$x = \frac{p_{11}\, z_1 + p_{12}\, z_2 + p_{13}\, z_3 + p_{14}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}},$$

$$y = \frac{p_{21}\, z_1 + p_{22}\, z_2 + p_{23}\, z_3 + p_{24}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}}.$$

---

11 degrees of freedom:

- location of the center (3 degrees of freedom);
- position of the image plane (3 degrees of freedom);
- choice of affine coordinates on the image plane up to overall scaling (5 degrees of freedom).

If the center of the projection is at ∞, then we have a parallel projection (the last row of P is $(0, 0, 0, 1)$)

$$
\begin{aligned}
x &= p_{11}\, z_1 + p_{12}\, z_2 + p_{13}\, z_3 + p_{14}, \\
y &= p_{21}\, z_1 + p_{22}\, z_2 + p_{23}\, z_3 + p_{24}.
\end{aligned}
$$

Relationship between the projection and the group-equivalence problems.

[Burdis, Hong and IK (2013)]

Proposition: Given $Z \subset \mathbb{R}^3$ and $X \subset \mathbb{R}^2$,

$\exists$ a central projection $P: Z \to X$

$$\Updownarrow$$

$\exists\, c_1, c_2, c_3 \in \mathbb{R}$ such that X is $PGL(3)$-equivalent to a planar curve:

$$Z_c = \left\{ \left(\frac{z_1 - c_1}{z_3 - c_3}, \quad \frac{z_2 - c_2}{z_3 - c_3} \right) \Big| (z_1, z_2, z_3) \in Z \right\}$$

$\mathbf{c} = (c_1, c_2, c_3)$ is the center of the projection.

Proof: $\boxed{P = AP_0B}$, where A is the left 3×3 submatrix of P,

$$P_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The $\boxed{\text{central projection problem}}$ can be reduced to a

$\boxed{\text{a } PGL(3)\text{-group-equivalence problem with 3 parameters}}$.

Similarly, the $\boxed{\text{parallel projection problem}}$ can be reduced to a

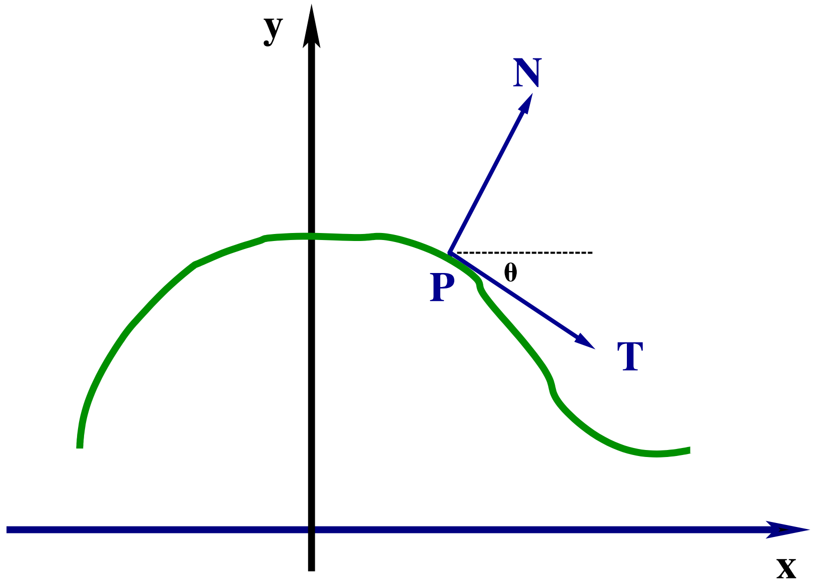
$\boxed{\text{a } A(2)\text{-group-equivalence problem with 2 parameters}}$.

$\Rightarrow$ algorithms for solving the projection problem.

The equivalence problem

starting with smooth parameterized curves under the SE(2)-action

Euclidean Curvature:

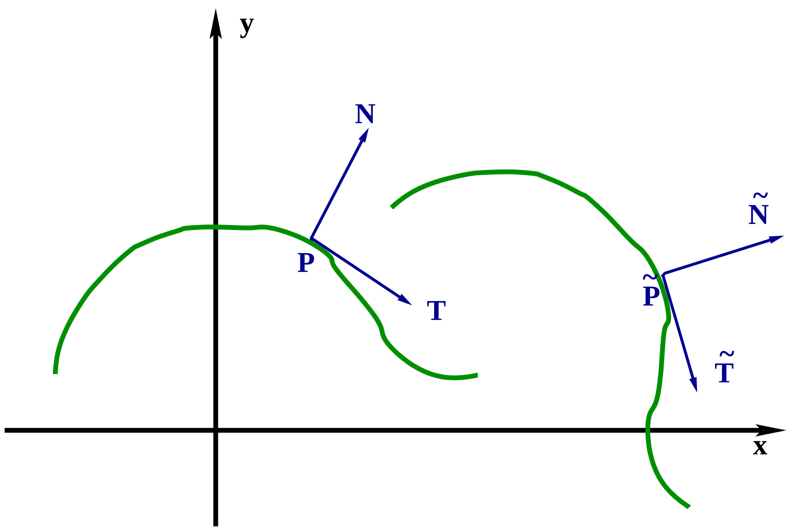


- $\gamma(t) = (x(t), y(t))$ at least C^2-smooth immersed curve.
- $\gamma'(t) = (x'(t), y'(t))$
- $|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
- $T = \frac{1}{|\gamma'|}\gamma' = (\cos\theta, \sin\theta)$
- θ is the signed angle between the x-axis and T.
- $N = \frac{1}{|\gamma'|}(-y', x') = (-\sin\theta, \cos\theta)$

- The arc-length parameter $s(t) = \int_{t_0}^{t} |\gamma'(\tau)| d\tau$
- Infinitesimal arc-length $\boxed{ds = |\gamma'(t)| dt}$.
- The arc-length derivative $\boxed{\frac{d}{ds} = \frac{1}{|\gamma'(t)|}\frac{d}{dt}}$.
- The signed Euclidean curvature

$$\boxed{\kappa = \frac{d\theta}{ds} = \frac{\det(\gamma'(t), \gamma''(t))}{|\gamma'(t)|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}}$$

Equivariant moving frames and invariants:



- $g \in SE(2) = SO(2) \ltimes \mathbb{R}^2$
- $\tilde{X} = g \cdot X, \quad \tilde{P} = g \cdot P$
- T, N are equivariant:
  $\tilde{T} = g \cdot T, \quad \tilde{N} = g \cdot N$
- κ is invariant: $\kappa_{\tilde{X}}(\tilde{P}) = \kappa_X(P)$

- Frenét equation (Frenét thesis 1847) $\boxed{\frac{dT}{ds} = \kappa N}$.
- $[T, N] \in SO(2), \quad P \in \mathbb{R}^2 \qquad \Longrightarrow \qquad (T, N, P) \in SE(2)$
- we have an equivariant map from {jets of curves} $\to SE(2)$.
- general definition of an equivariant moving frame map [by Fels and Olver, 1999] $\Rightarrow$ a wide range of applications.

- Frenét equations: $\begin{bmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dP}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ P \end{bmatrix}$

- For sufficiently smooth curves,
  $\kappa_s = \frac{d\kappa}{ds}, \kappa_{ss} = \frac{d^2\kappa}{ds^2}, \kappa_{sss}, \ldots$ - are higher order invariants.

- Any differential invariants, i. e. smooth invariant functions built from $x, y, x_t, y_t, x_{tt}, y_{tt}, \ldots$ is a function of those.

A function $\mathbb{R}^m \to \mathbb{R}^n$ is C^0-smooth if it is continuous, C^k-smooth if all its (partial) derivatives up to order k are continuous, C^∞-smooth if the derivatives of any order are continuous, and C^ω if it is analytic.

Reconstructing a curve from its curvature:

Proposition: $\forall$ continuous $\kappa\colon \mathbb{R} \to \mathbb{R}$, $\exists!$ curve $X \subset \mathbb{R}^2$ with a C^2-smooth arc-length parameterization $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, such that $\gamma(0) = (0,0)$, $\gamma'(0) = [1,0]^T$, and $\kappa(s)$ is the curvature of X at the point $\gamma(s)$.

Proof:

- $\kappa = \frac{d\theta}{ds} \quad \Longrightarrow \quad \boxed{\theta(s) = \int_0^s \kappa(\tau)d\tau.}$

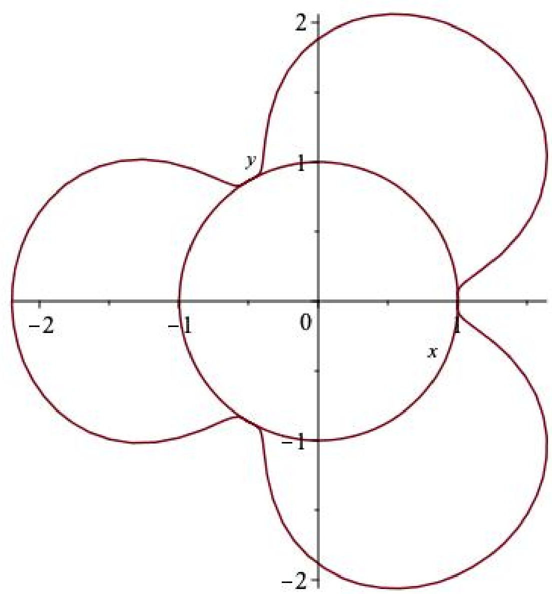
- $\frac{d\gamma}{ds} = (\cos\theta, \sin\theta) \quad \Longrightarrow \quad \boxed{\gamma(s) = (\int_0^s \cos\theta(\tau)d\tau,\ \int_0^s \sin\theta(\tau)d\tau)}$

Congruence in terms of curvature:

Proposition:Let γ_1 and γ_2 be arc-length parameterizations of curves X_1 and X_2 and κ_1, κ_2 be the corresponding curvature functions. Then

- $\exists c \in \mathbb{R}$, such that $\kappa_1(s) = \kappa_2(s+c) \quad \Longrightarrow \quad X_1 \underset{SE(2)}{\cong} X_2$.

- $\Longleftarrow$ is true if X_1 and X_2 are simple.



Solution based on curvature as a function of the arc-length:

Given two simple planar curves X_1 and X_2,

1. Compute their arc-length parameterizations $\gamma_1(s)$ and $\gamma_2(s)$;

2. Compute their curvatures $\kappa_1(s)$ and $\kappa_2(s)$;

3. $X_1 \underset{SE(2)}{\cong} X_2 \quad \Longleftrightarrow \quad \exists c \in \mathbb{R}$ such that $\kappa_1(s) = \kappa_2(s+c)$

Drawbacks:

- step 1 - computing the arc-length parameterization is difficult;

- step 3 - to avoid the shift we must match “the initial point”.

The differential signature (κ, κ_s) is a parametrization independent invariant:

Calabi, Olver, Shakiban, Tannenbaum, Haker (1998)

Definition: Let $X \subset \mathbb{R}^2$ be an immersed, at least C^3-smooth curve.

- for a parameterization $\gamma \colon I \subset \mathbb{R} \to X$, a parameterized signature map $\sigma_\gamma \colon I \to \mathbb{R}^2$ is defined by $\boxed{\sigma_\gamma(t) = (\kappa(t), \kappa_s(t))}$, where

  $\boxed{\kappa = \frac{\det(\gamma', \gamma'')}{|\gamma'|^3} \text{ and } \kappa_s = \frac{(\gamma' \cdot \gamma') \det(\gamma', \gamma''') - 3(\gamma' \cdot \gamma'') \det(\gamma', \gamma'')}{|\gamma'|^6}}$

- the signature (the signature set) of X is the image of this map:

  $\boxed{S_X = \mathrm{Im}\sigma_\gamma;}$

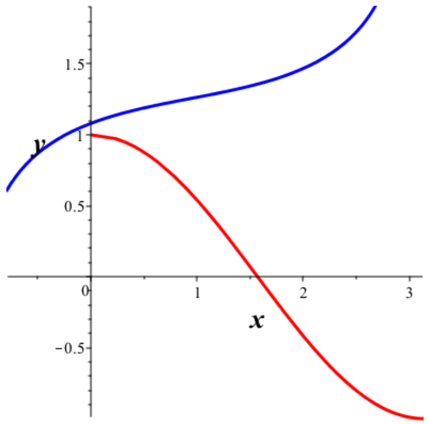
Proposition:

- S_X does not depend on parameterization.
- S_X is $SE(2)$ invariant: $\boxed{X_1 \underset{SE(2)}{\cong} X_2 \implies S_{X_1} = S_{X_2}}$.

Example:

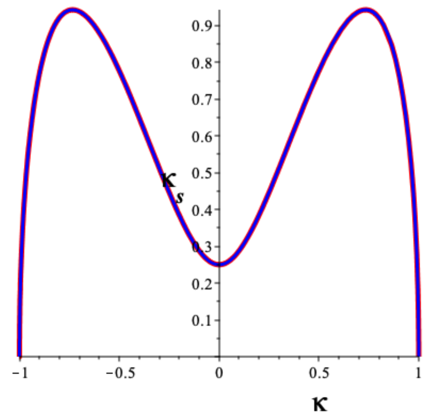
$\gamma_1(t) = \left(t^3,\, \cos(t^3)\right),$ $t \in [0, \pi^{\frac{1}{3}}]$	$\gamma_2(t) = (\frac{3}{5}t - \frac{4}{5}\cos t,\, \frac{4}{5}t + \frac{3}{5}\cos t),$ $t \in [0, \pi]$
$\kappa(t) = -\frac{\cos(t^3)}{(\sin^2(t^3)+1)^{\frac{3}{2}}}$	$\kappa(t) = -\frac{\cos t}{(\sin^2 t+1)^{\frac{3}{2}}}$
$\kappa_s(t) = \frac{2\sin(t^3)(\cos^2(t^3)+1)}{(\sin^2(t^3)+1)^3}$	$\kappa_s(t) = \frac{2\,\sin(t)(\cos^2 t+1)}{(\sin^2 t+1)^3}$

$X_1 = Im(\gamma_1)$ and $X_2 = Im(\gamma_2)$



$X_1 \underset{SE(2)}{\cong} X_2$

The signatures are equal:



$S_{X_1} = S_{X_2}$

Observations:

- $\boxed{\kappa \equiv 0}$ on X $\iff$ X is a line (or a line segment) $\iff$ S_X is a point $(0, 0)$.

- $\boxed{\kappa \equiv c \neq 0}$ on X $\iff$ X is a circle (or a circular arc) $\iff$ S_X is a point $(c, 0)$.

- $\boxed{\text{otherwise}}$ S_X is one-dimensional phase portrait: if $\gamma(s)$ is an arc-length parameterization of X, then $(\kappa(s), \kappa'(s))$ is "in-phase" parametrization of S_X.

A fundamental question:

Do signatures characterize equivalence classes of C^3 smooth curves?

Signatures are invariant:

$$X_1 \underset{SE(2)}{\cong} X_2 \qquad \boxed{\Longrightarrow} \qquad S_{X_1} = S_{X_2}$$

but are they separating?

$$S_{X_1} = S_{X_2} \qquad \boxed{\overset{?}{\Longrightarrow}} \qquad X_1 \underset{SE(2)}{\cong} X_2 \quad .$$

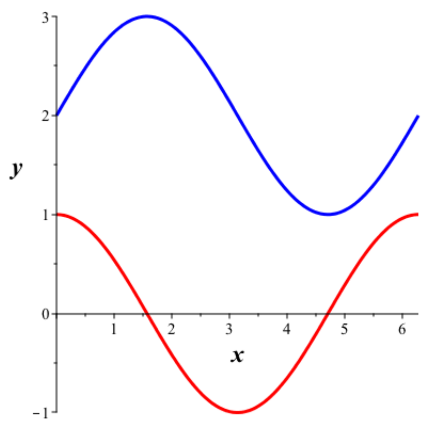
Answer: No, unless we put some additional restrictions on a class of curves we consider, or augment the signature with some additional information.

Non-congruent curves with the same signatures

I. Non-congruent curve segments

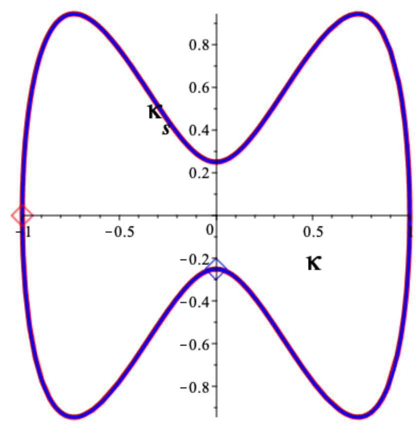
$\gamma_1(t) = (t,\, \cos t),\ \ t \in [0, 2\pi]$	$\gamma_2(t) = (t, \sin t + 2)\ \ t \in [0, 2\pi],$
$\kappa(t) = -\frac{\cos t}{(\sin^2 t+1)^{\frac{3}{2}}}$	$\kappa(t) = -\frac{\sin t}{(\cos^2 t+1)^{\frac{3}{2}}}$
$\kappa_s(t) = \frac{2 \sin t(\cos^2 t+1)}{(\sin^2 t+1)^3}$	$\kappa_s(t) = -\frac{2\ \cos t(\sin^2 t+1)}{(\cos^2 t+1)^3}$

$X_1 = Im(\gamma_1)$ and $X_2 = Im(\gamma_2)$



$$X_1 \underset{SE(2)}{\not\cong} X_2$$

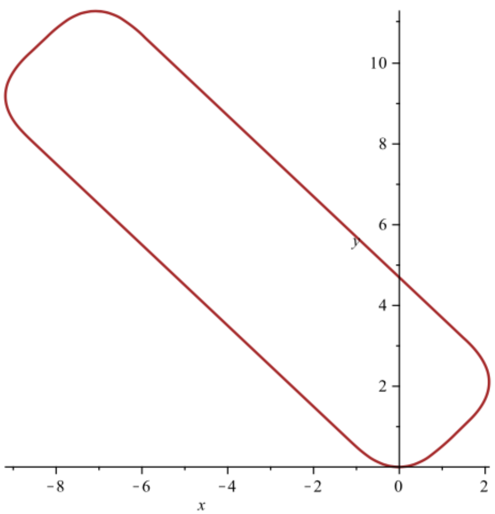
The signatures are equal:

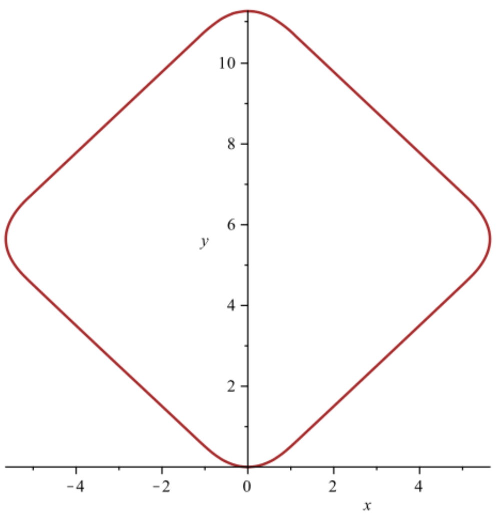


$$S_{X_1} = S_{X_2}$$

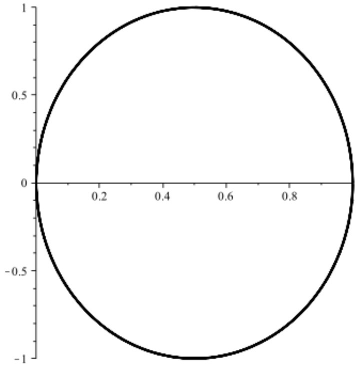
II. Non-congruent closed curves containing straight segments and circular arcs (degenerate curves .)

The following two curves





share the same signature:



Images are from Hickman (2012)

Can we construct examples of non-congruent curves, such that:

- whose parametrization defined over $\mathbb{R}$ ($\gamma \colon \mathbb{R} \to \mathbb{R}^2$);
- which do not contain segments of constant curvature (non-degenerate curves)?

In addition we may want them to be

- closed (γ is periodic);
- simple.

III. Non-congruent $\boxed{\text{non-degenerate}}$ closed curves with the same signatures.

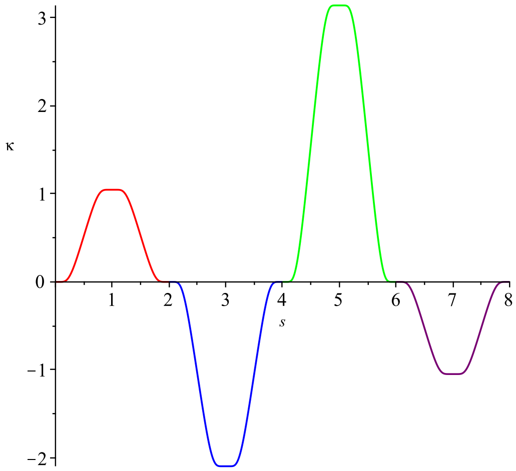
Geiger and IK (2020):

- Describe a general mechanism for constructing families of non-congruent curves with the same signature.

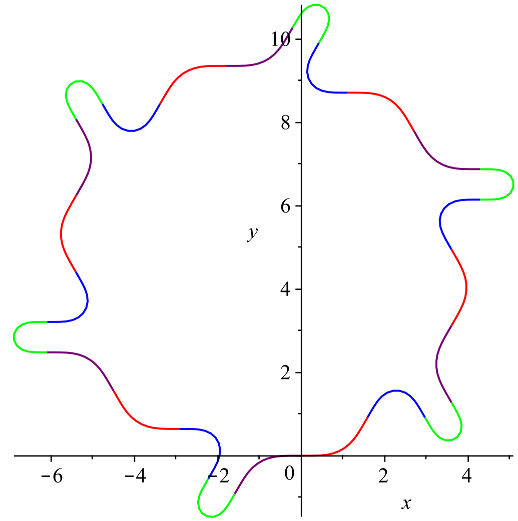
- Introduce the notion of a signature quiver and use it to
  - to formulate congruence criteria for non-degenerate curves.
  - encode global and local symmetries of curves.

Our results continue the line of research in

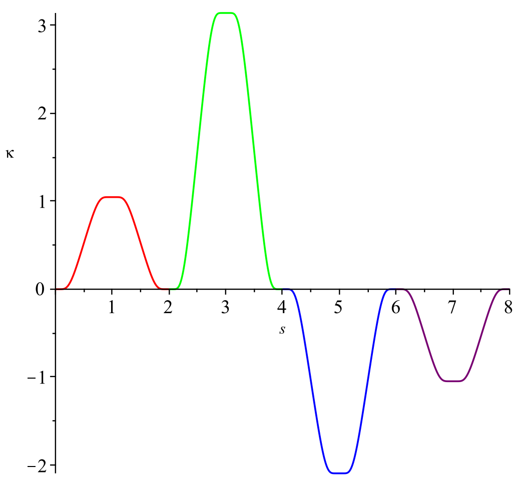
- Musso and Nicolodi (2009)
- Hickman (2012)
- Olver (2016)



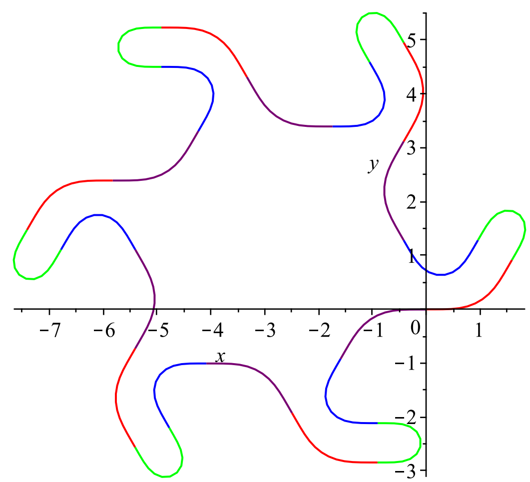
Curvature $\kappa_1(s)$.



X_1 is reconstructed from the periodic continuation of $\kappa_1(s)$

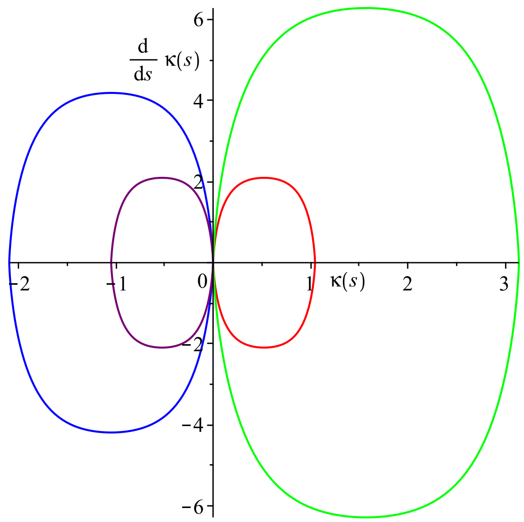


Curvature $\kappa_2(s)$.

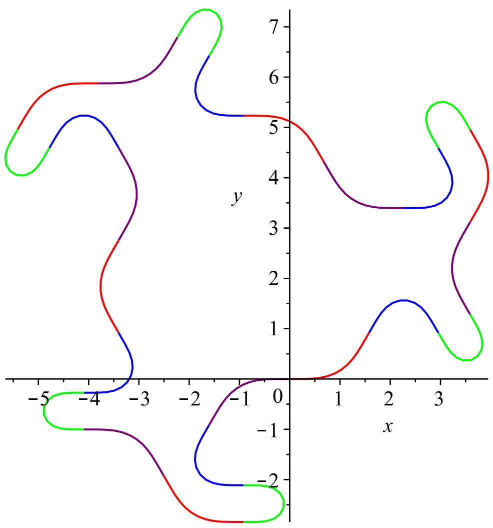


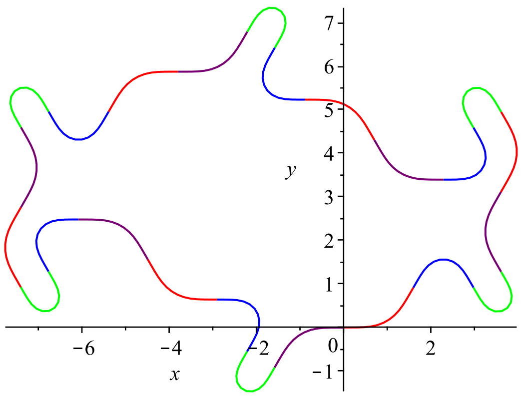
X_2 is reconstructed from the periodic continuation of $\kappa_2(s)$.

X_1 and X_2 are non-congruent, but have the same signature:



More non-congruent curves with the same signature:





X_3 has the symmetry group $\mathbb{Z}_3$ X_4 has the symmetry group $\mathbb{Z}_2$.

So, what does equality of the signatures $S_{X_1} = S_{X_2}$ imply?

Local congruence: $\forall p \in X_1$ such that $\kappa_s(p) \neq 0$, $\exists$ a curve piece $\hat{X}_1 \subset X_1$, $p \in \hat{X}_1$, which is $SE(2)$-congruent to a curve piece $\hat{X}_2 \subset X_2$. [Calabi, Olver, Shakiban, Tannenbaum, Haker (1998)]

Global congruence, $X_1 \underset{SE(2)}{\cong} X_2$, when:

- X_1 and X_2 have analytic parameterizations with domain $\mathbb{R}$;

- X_1 and X_2 are algebraic;

  (We will discuss an adaptation of the differential signature to algebraic curves later)

- X_1 and X_2 have no vertices [Hoff and Olver, 2013];

- X_1 and X_2 are non-degenerate closed with simple signatures [Geiger and IK, 2020].

Differential signatures for other groups

Classifying Differential Invariants

Theorem: Let a Lie group $G \curvearrowright \mathbb{R}^2$, $\dim G = r$. Consider the induced action on the set of sufficiently smooth parameterized curves $\gamma(t) = (x(t), y(t))$. Then

- $\exists$ classifying differential invariants K_1 and K_2 of orders* k and r, respectively, $k < r$, s. t. the (K_1, K_2)-signature can be used to solve the local equivalence problems for sufficiently regular planar curves under G-action.

- For most actions† the signature is a phase portrait:

  - $\boxed{K_1 = \kappa_G}$ is the $(r-1)$-th order differential invariant, called $\boxed{G\text{-curvature}}$.

  - $\boxed{K_2 = \frac{d\kappa_G}{ds_G}}$ is the derivative of the $\boxed{G\text{-curvature}}$ with respect to the $\boxed{G\text{-arc-length}}$.

*The order of a differential invariants is the highest order of the derivatives of γ it depends on.

†See Theorem 5.24, Olver, "Equivalence, Invariance and Symmetries" (1995).

Curvatures and arc-lengths for $SE(2) \subset SA(2) \subset PGL(3)$

G	G-curvature	G-arc-length
$SE(2)$	$\kappa = \frac{x'y''-y'x''}{(x'^2+y'^2)^{\frac{2}{3}}}$	$ds = \sqrt{x'^2+y'^2}dt$
$SA(2)$	$\mu = \frac{3\,\kappa\,(\kappa_{ss}+3\,\kappa^3)-5\,\kappa_s^2}{9\,\kappa^{8/3}}$	$d\alpha = \kappa^{1/3} ds$
$PGL(3)$	$\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha-7\,\mu_{\alpha\alpha}^2-9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}$	$d\rho = \mu_\alpha^{1/3} d\alpha$

[Inductive and recursive constructions of moving frames and invariants: Kogan (2003), Olver (2018), Olver and Valiquette (2018)]

Differential signatures of algebraic curves

Rational Classifying Differential Invariants

Theorem: [Ruddy, Vinzant and IK (2020)] Let a complex* algebraic group $G \subset PGL(3) \curvearrowright \mathbb{C}^2$, $\dim G = r$. Consider the induced action on the set of algebraic curves defined by an irreducible polynomial $F(x, y) = 0$. Then

- $\exists$ classifying differential invariants K_1 and K_2, depending rationally on the derivatives of F of orders k and r, respectively, $k < r$, such that (K_1, K_2)-signature can be used to solve the global G-equivalence problems for generic algebraic curves.

---

*Real case is more subtle.

In more details:

Let $\mathcal{P}_d = \{F(x,y) | \deg F \le d\}$ such that $\binom{d+2}{2} - 2 \ge \dim G$. There exists a Zariski closed subset of $\mathcal{P}_d$, such that for all curves whose defining equation lies outside of this set:

- $\boxed{\text{the signature map of } X}$, $\sigma_X = (K_1|_X, K_2|_X)\colon X - - > \mathbb{C}^2$, is a rational map;

- $\boxed{\text{the signature of } X}$, $S_X = \overline{\mathrm{Im}(\sigma_X)}$, is
  - a point $\Longleftrightarrow$ the symmetry group of X has positive dimension;
  - an algebraic curve, whose defining polynomial (in principle) can be computed by an elimination algorithm;

- σ_X is generically $n : 1$, where $\boxed{n \text{ is the cardinality of the symmetry group of } X}$;

- $\boxed{X_1 \underset{G}{\cong} X_2 \quad \Longleftrightarrow S_{X_1} = S_{X_2}}$.

Examples of rational classifying invariants

G	$SE(2)$	$E(2)$	$SA(2)$	$A(2)$	$PGL(3)$
K_1	κ^2	κ^2	μ^3	$\frac{\mu_\alpha^2}{\mu^3}$	η^3
K_2	κ_s	κ_s^2	μ_α	$\frac{\mu_{\alpha\alpha}}{\mu^2}$	η_ρ
irreducible exceptional	lines		lines and conics		

In general, K_1 and K_2 can be computed by

1. Prolongations of group actions.
2. Algorithm for computing a generating set of rational invariants [Derksen (1999), Hubert and IK (2007), Derksen and Kemper (2015)].

Are there other classifying pairs?

$$\tilde{K}_1 = \frac{aK_1 + b}{cK_1 + d} \text{ and } \tilde{K}_2 = \frac{\alpha(K_1)K_2 + \beta(K_1)}{\gamma(K_1)K_2 + \delta(K_1)},$$

where

$$a, b, c, d \in \mathbb{C} \text{ and } \alpha, \beta, \gamma, \delta \in \mathbb{C}[K_1].$$

Degree of a signature

Theorem: Signatures of generic curves of degree d have the same degree and this degree is the upper bound.

G	$SE(2)$	$E(2)$	$SA(2)$	$A(2)$	$PGL(3)$
deg S_X	$6d^2-6d$	$12d^2-12d$	$24d^2-48d$	$24d^2-48d$	$96d^2-216d$

Signatures of symmetric curves have much lower degree

Example: Fermat curves X_d, defined by $\boxed{F_d(x,y)=x^d+y^d+1}$ of degree $d>2$, under $PGL(3)$-action.

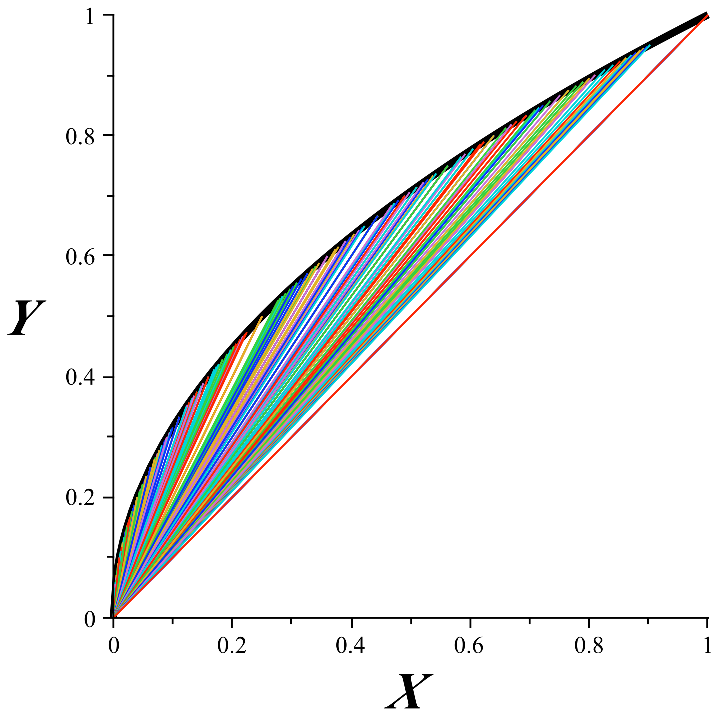
- the symmetry group is $S_3 \rtimes (\mathbb{Z}_d \times \mathbb{Z}_d)$ of cardinality $6d^2$;
- the signature of has degree 4 independently of d.

Its defining polynomial is

$$
\begin{aligned}
&49392(d-2)^4d^3(d+1)^4(2d-1)^4K_2^4+602112(d-2)^4d^3(d+1)^4(2d-1)^4K_1K_2^2\\
&+10584(d-2)^3d^2(d+1)^3(2d-1)^3\left(10d^2-3d+3\right)\left(34d^2-27d+27\right)K_2^3\\
&+1835008(d-2)^4d^3(d+1)^4(2d-1)^4K_1^2-9289728(d-2)^3d^2(d+1)^3(2d-1)^3\left(d^2-d+1\right)^2K_1K_2\\
&+61236(d-2)^2d(d+1)^2(2d-1)^2\left(d^2-d+1\right)\left(10d^2-3d+3\right)^2\left(16d^2-9d+9\right)K_2^2\\
&-23328(d-2)^2d(d+1)^2(2d-1)^2\left(11792d^8-17376d^7+28152d^6-24424d^5+19473d^4-8940d^3\right.\\
&\left.+3358d^2-324d+81\right)K_1+118098(d-2)(d+1)(2d-1)\left(d^2-d+1\right)^2\left(10d^2-3d+3\right)^4K_2\\
&+531441d\left(d^2-d+1\right)^3\left(10d^2-3d+3\right)^4.
\end{aligned}
$$

Integral Invariants

Example: Signed area between the curve and a secant



$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b]$$

$$X(t) = x(t) - x(a)$$
$$Y(t) = y(t) - y(a)$$

$$\boxed{I^{[0,1]}(t) = \int_a^t Y(\tau)\, dX(\tau) - \frac{1}{2} X(t)\, Y(t)}$$

Question: *With respect to which transformation $I^{[0,1]}$ is invariant ?*

Answer: special affine -$SA(2) \Longrightarrow$ special Euclidean - $SE(2)$.

How to obtain integral invariants for planar curves?

$$\boxed{\gamma(t) = (x(t), y(t)),\ t \in [0,1]}$$

1. Shift starting point to the origin: $X(t) = x(t) - x(0)$, $Y(t) = y(t) - y(0)$,

2. Prolong the action to integral variables:

$$\boxed{X^{[i,j]}(t) = \int_0^t X(\tau)^i\, Y(\tau)^j\, dX(\tau).}$$

3. Fels-Olver m.-f. method $\Rightarrow$ Invariants.

Examples:

0-th order $r = \sqrt{X^2 + Y^2}$ - $E(2)$-invariant

1-st order $I^{[0,1]} = X^{[0,1]} - \frac{1}{2} X\, Y$ -$SA(2)$ and $SE(2)$-invariant.

2-nd order

- $I^{[1,1]} = Y\, X^{[1,1]} - \frac{1}{2} X\, X^{[0,2]} - \frac{1}{6} X^2\, Y^2$-
  $SA(2)$ and $E(2)$-invariant
- $I^{[0,2]} = Y\, X^{[0,2]} + 2X\, X^{[1,1]} - \frac{1}{3} X\, Y^3 - \frac{2}{3} X^3\, Y$
  E_2-invariant

...

[Hann and Hickman (2002) - planar curves; Feng, Krim and IK (2010) - inductive formulas, space curves]

Integral signatures for planar curves

- $SE(2)$-signature $\left(r,\ I^{[0,1]}\right)$

- $E(2)$- signatures $\left(r,\ \left(I^{[0,1]}\right)^2\right)$ or $\left(r,\ I^{[1,1]}\right)$.

- similarity signature: $\left(\dfrac{\left(I^{[0,1]}\right)^2}{r^4},\ \dfrac{I^{[1,1]}}{r^4}\right)$

- $SA(2)$-signature $\left(I^{[0,1]},\ I^{[1,1]}\right)$

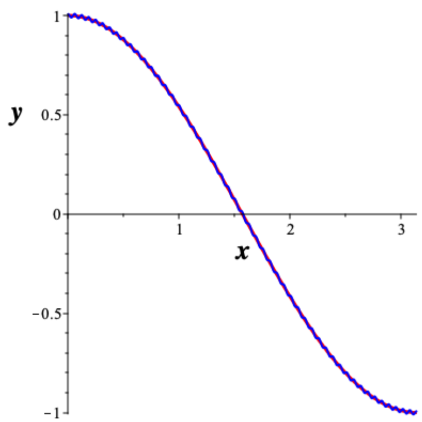
Main features of signatures based on integral invariants:

- Stability with respect to noise and high frequency small perturbations.
- Easily derived invariant numerical approximations.
- Dependence on initial point (semi-local).
- Local, independent of initial point, discrete integral signatures can be defined [Feng, Krim and IK (2010)]

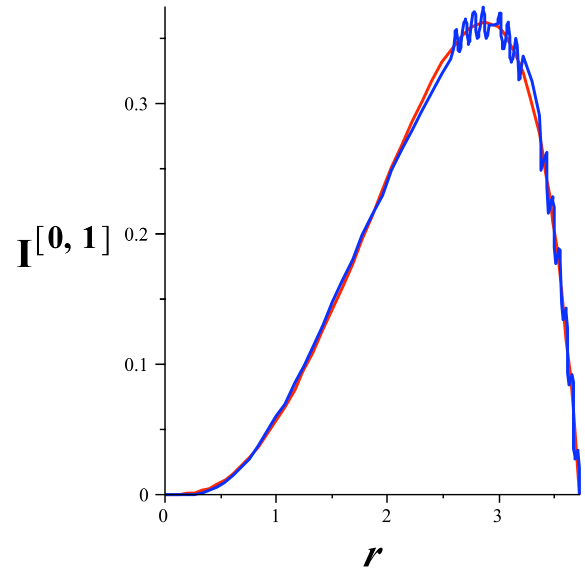
Stability of integral signature

$\gamma(t) = (t, \cos t)$, $t \in [0, \pi]$	$\tilde{\gamma}(t) = (t, \cos(t) + \frac{1}{100}\sin(100\,t)$, $t \in [0, \pi]$

Images of γ and $\tilde{\gamma}$ in $\mathbb{R}^2$

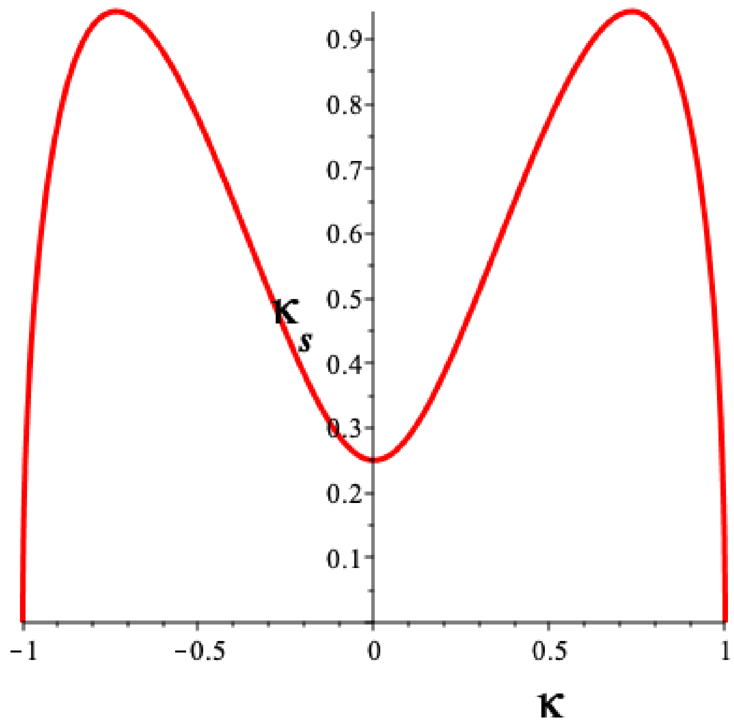


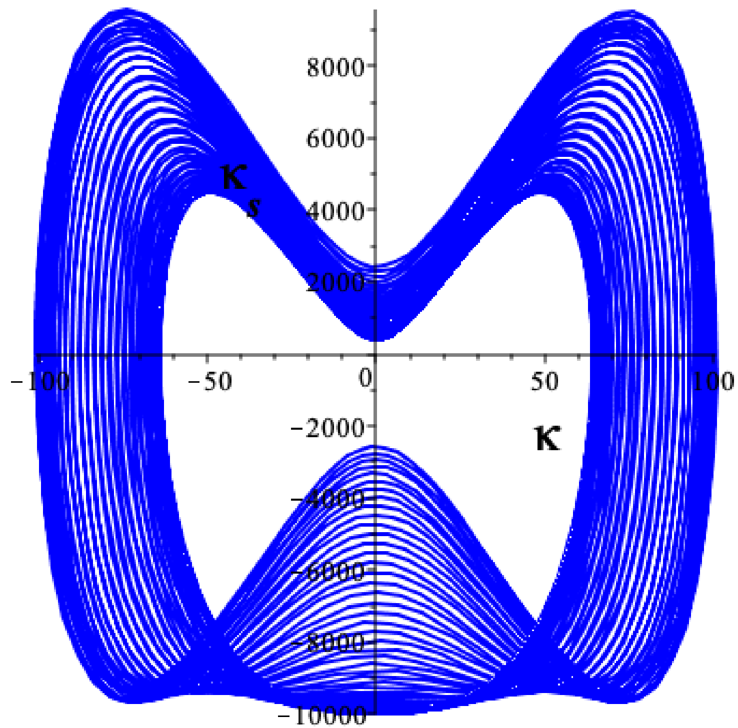
$SE(2)$- signatures for γ and $\tilde{\gamma}$



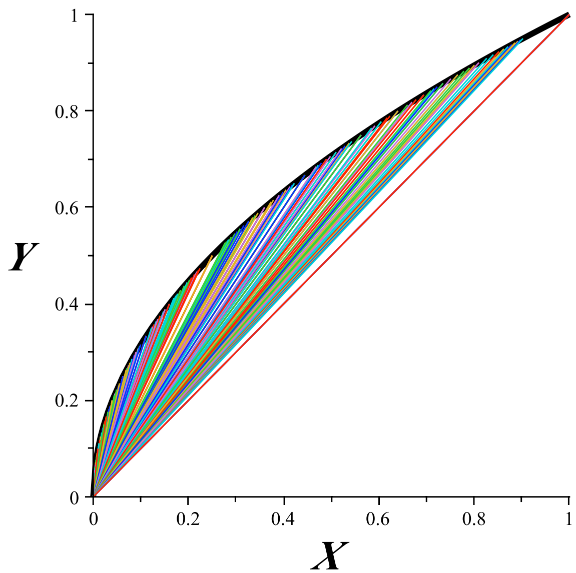
$SE(2)$-differential signatures for the same curves

$\gamma(t) = (t, \cos t)$, $t \in [0, \pi]$	$\tilde{\gamma}(t) = (t, \cos(t) + \frac{1}{100} \sin(100\, t))$, $t \in [0, \pi]$



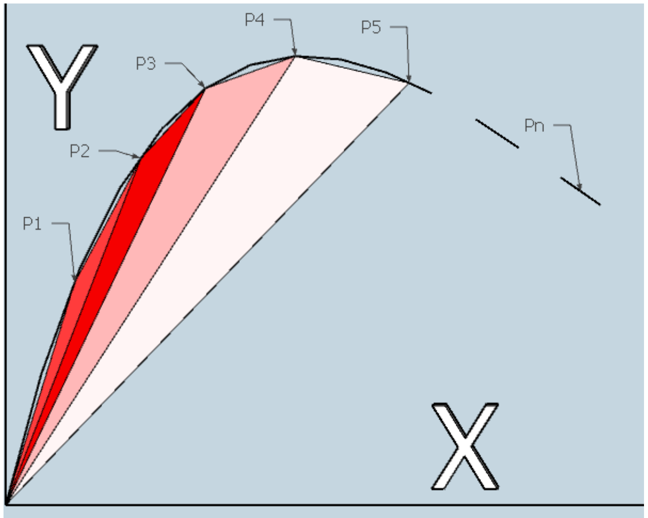


Numerical approximations:



$$\begin{aligned}\gamma(t) &= (x(t), y(t)), \ t \in [a,b] \\ X(t) &= x(t) - x(a) \\ Y(t) &= y(t) - y(a)\end{aligned}$$

$$\boxed{I^{[0,1]}(t) = \int_a^t Y(\tau)\, dX(\tau) - \tfrac{1}{2} X(t)\, Y(t)}$$



$$\begin{aligned}\gamma(n) &= (x(n), y(n)), \ n = 0, 1, 2, \ldots \\ X(n) &= x(n) - x(0) \\ Y(n) &= y(n) - y(0)\end{aligned}$$

$$\boxed{I^{[0,1]}(n) = \frac{1}{2} \sum_{i=1}^{n} \begin{vmatrix} X_i & X_{i-1} \\ Y_i & Y_{i-1} \end{vmatrix}}$$

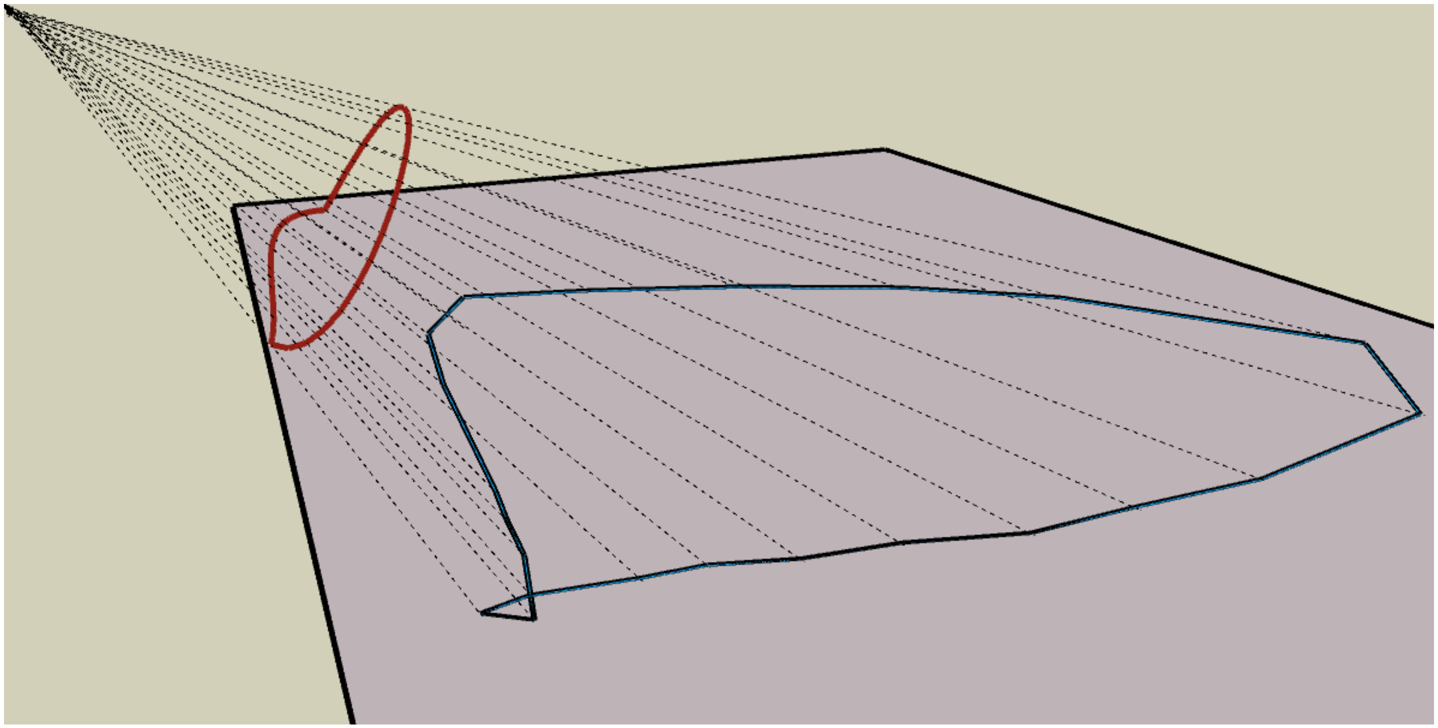
Applications of integral invariants

- Feng, Krim, and IK (2007) (face recognition)
- Golubitsky, Mazalov and Watt (2009) (handwriting recognition)
- Katie Iwancio Ph.D. Thesis, NCSU (2009) (contour-matching)
- Susan Crook Ph.D. Thesis, NCSU (2013) (puzzles, handwriting)

Returning to the projection problem:

Relations between invariants of an object and its image

Relations between invariants of an object and its image



Invariants with respect to which group-action on $\mathbb{R}^3$? on $\mathbb{R}^2$?

- on $\mathbb{R}^3$ - standard linear action of $GL(3)$(centro-affine invariants) or $SL(3)$-action (centro-equi-affine invariants)

- on $\mathbb{R}^2$ - projective action (projective invariants)

Centro-equi-affine invariants for space curves in terms of the invariants of the planar images:

Theorem: [Olver and IK (2015)]

Differential algebra of centro-equi-affine invariants of space is generated by:

- $\widehat{\eta} = P_0^*(\eta)$
- $\zeta = z_3 \, P_0^* \left(\frac{1}{\mu_\alpha^{1/3}} \right)$
- $d\widehat{\rho} = P_0^*(d\rho)$,

where

- η and $d\rho$ are planar projective curvature and arc-length;
- μ and $d\alpha$ are planar equi-affine curvature and arc-length;
- P_0 is the standard central projection $x = \frac{z_1}{z_3}$, $x = \frac{z_2}{z_3}$ from the origin to the plane $z_3 = 1$:

Centro-equi-affine curvature, torsion and arc-lengths: [Olver (2010)]

Let $\mathcal{Z} \subset \mathbb{R}^3$ be parametric curve $\mathbf{z}(t) = \big(z_1(t),\, z_2(t),\, z_3(t)\big)$, then

- centro-equi-affine arc-lengths $dS := |\mathbf{z},\, \dot{\mathbf{z}},\, \ddot{\mathbf{z}}|\; dt$ (undefined when $\mathcal{Z}$ is contained in the plane spanned by $\mathbf{z}(0)$ and $\dot{\mathbf{z}}(0)$).

- centro-equi-affine torsion $\tau = |\mathbf{z}_S,\, \mathbf{z}_{SS},\, \mathbf{z}_{SS}|$ ($\tau \equiv 0 \iff \mathcal{Z}$ is coplanar).

- centro-equi-affine curvature $\kappa = |\mathbf{z},\, \mathbf{z}_{SS},\, \mathbf{z}_{SS}|$

Theorem κ, τ and dS generate differential algebra of centro-affine invariants.

Relationship between two generating sets: [Olver and IK (2015)]

- $\widehat{\eta} = \dfrac{a_{ss}\, a - \frac{7}{6}\, a_s^2 - \frac{3}{2}\, \kappa\, a^2}{3^{2/3}\, a^{8/3}}$;
- $\zeta = (3a)^{-1/3}$;
- $d\widehat{\rho} = (3\, a)^{1/3}\, dS$;

where $a = \kappa_S + 2\,\tau$ is identically zero iff $P_0(\mathcal{Z})$ is a line or a conic.

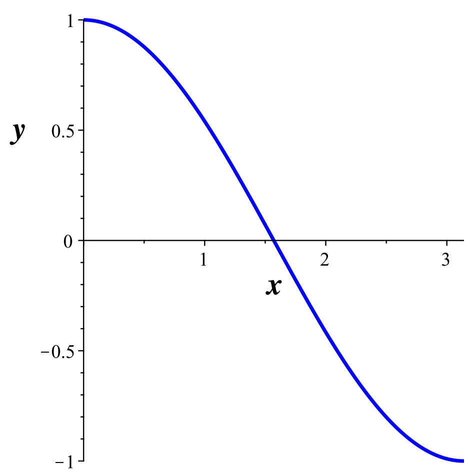
Additional slides

Warning: $\boxed{\kappa(t) \text{ for an arbitrary parametrization can not be used}}$ $\boxed{\text{to solve the equivalence problem!}}$

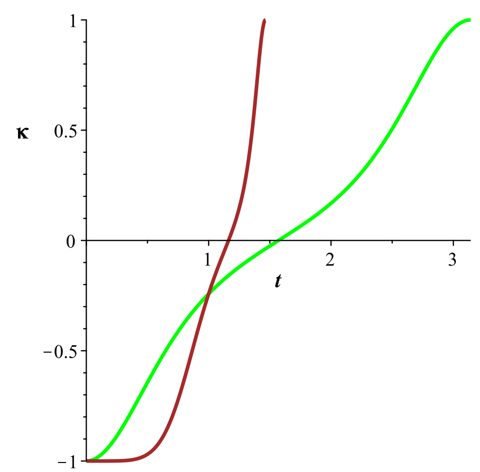
Example: Consider two parametrization of $y = \cos(x)$, $\quad x \in [0, \pi]$:

$$\gamma(t) = (t, \cos t),\ t \in [0, \pi] \text{ and } \tilde{\gamma}(t) = (t^3, \cos(t^3)),\ t \in [0, \pi^{\frac{1}{3}}].$$

The corresponding graphs $\kappa(t) = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{\frac{2}{3}}}$ are not related by a shift:



The graph of $y = \cos(x)$ for $x \in [0, \pi]$.



The graphs of $\kappa(t)$ for $\gamma(t)$ and $\tilde{\gamma}(t)$.

Non-degeneracy and vertices

Definition:

- Let $\gamma(t)$ be a parameterization of X. Then a point $p = \gamma(t_0)$ is a $\boxed{\text{vertex}}$ if $\kappa_s(t_0) = 0$.
- X with a finite set of vertices is called $\boxed{\text{non-degenerate}}$.

Observations:

- Curves containing circular arcs or straight segments are $\boxed{\text{degenerate}}$.
- Under the signature map, vertices are mapped to the horizontal axis.
- Every closed curve has at least 2 vertices and a simple closed curve has at least 4. Mukhopadhyaya (1909), Kneser (1912), AMS Notices overview DeTurck, Gluck, Pomerleano, and Shea Vick (2007)

Theorem. [Musso and Nicolodi (2009)] *Any closed phase portrait is the Euclidean signature of a 1-parameter family of non congruent closed curves.*

Families of curves constructed in the proof of this theorem contained at most one non-degenerate curve.

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