Geometric Methods in Computer Vision

Jose Agudelo, Brooke Dippold, Ian Klein, Alex Kokot

North Carolina State University

NC STATE UNIVERSITY

Abstract

In high school geometry, two shapes are congruent if they are related by rotations, translations, and reflections. These transformations comprise the Euclidean group, but this is not the only way to define congruence. Congruence can be defined by a variety of transformation groups. Motivated by applications in automated assembly, image recognition, and arm movements, we studied the special Euclidean and equi-affine notions of congruence. One method for curve recognition relies on invariant properties of curves, like curvatures and their derivatives. We have reconstructed curves from their curvatures using power series for analytic curvatures and Picard iterations for non-analytic curvatures. In application, curves are given approximately, so we examined relationships between closeness of invariants and the closeness of the curves reconstructed from these invariants.

Reconstruction from Equi-Affine Curvature

Given a curve, γ, we know that its tangent vector, T, and its equi-affine curvature, μ, are related by the following differential equation [1]:

$$T_{\alpha\alpha} = \mu(\alpha)T \quad (1)$$

Where α denotes the equi-affine arclength. When $\mu(\alpha)$ is *constant*, (1) is a constant coefficient linear system of equation. When $\mu > 0$, γ is a hyperbola. When $\mu < 0$, γ is an ellipse. When $\mu = 0$, γ is a parabola. Here we introduce a power series method to reconstruct γ when $\mu(\alpha)$ is *analytic* and a Picard Iteration method when $\mu(\alpha)$ is *non-analytic*.

Power Series Method

Each component of (1) can be written as a power series.

$$T = b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \cdots + b_n\alpha^n + \ldots$$

$$T\alpha\alpha = 2b_2 + 3\cdot 2b_3\alpha + \cdots + n(n-1)b_n\alpha^{(n-2)} + \ldots$$

When $\mu(\alpha)$ is an *analytic* function, it can be represented as a power series with known vector coefficients. We substitute these series into (1) and solve for the unknown vector coefficients b_n. The values for b_0 and b_1 are initial conditions determined by the tangent and normal vectors when $\alpha = 0$. The solution tangent vector can be integrated to find the solution γ vector, an estimation of the curve with curvature $\mu(\alpha)$.

As an example, the solution for the tangent vector of γ when $\mu(\alpha) = \alpha^k$ where $k \in \mathbb{Z}$ and $k \geq 1$ is given below.

$$T = b_0 + b_1\alpha + \sum_{j=1}^{\infty}\left(\left(b_0\cdot\prod_{i=1}^{j}\frac{1}{(k+2)i\cdot((k+2)i-1)}\right) + \left(b_1\cdot\prod_{i=1}^{j}\frac{1}{(k+2)i\cdot((k+2)i+1)}\right)\alpha\right)\alpha^{(k+2)j}$$

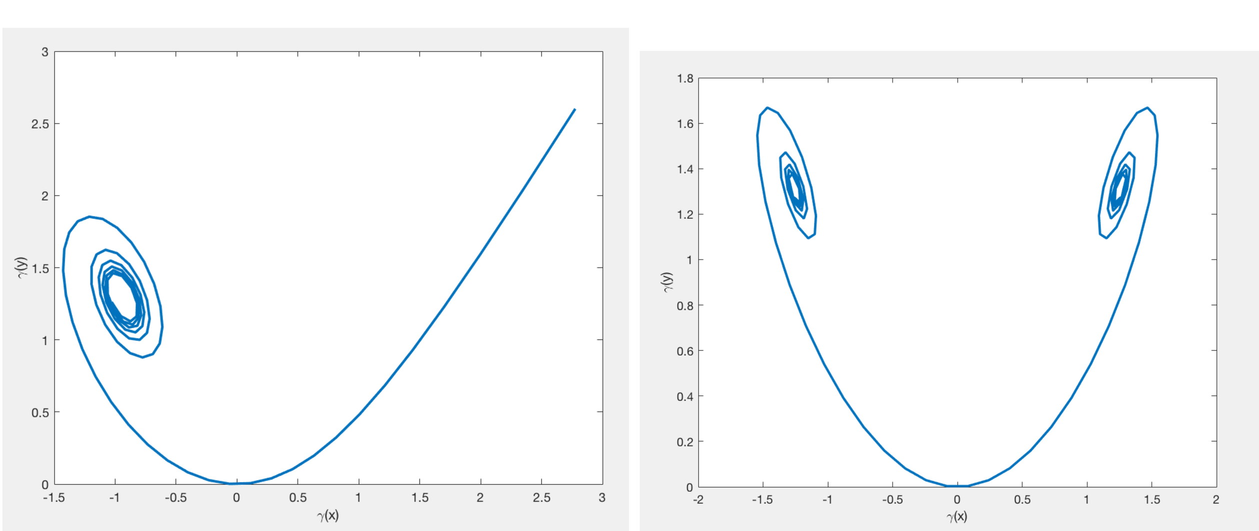


Fig. 1: Left: $\gamma(\alpha)$ defined by $\mu(\alpha) = \alpha$ Right: $\gamma(\alpha)$ defined by $\mu(\alpha) = -\alpha^2$

Picard Iteration Method

Using Picard iterations, we can find or approximate the tangent vector, T, of the curve, γ with equi-affine curvature $\mu(\alpha)$ through the iterative process:

$$\begin{bmatrix}T(\alpha)\\N(\alpha)\end{bmatrix}_k = \begin{bmatrix}T(\alpha_0)\\N(\alpha_0)\end{bmatrix} + \int_{\alpha_0}^{\alpha}\begin{bmatrix}0 & 1\\ \mu(\sigma) & 0\end{bmatrix}\begin{bmatrix}T(\sigma)\\N(\sigma)\end{bmatrix}_{k-1} d\sigma$$

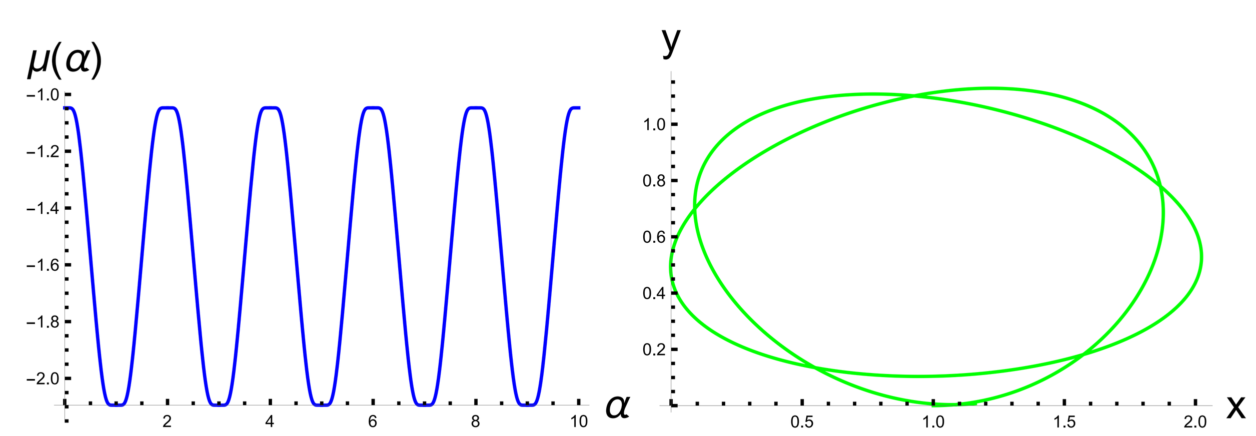
$$\text{where, } \begin{bmatrix}T(\alpha)\\N(\alpha)\end{bmatrix}_0 = \begin{bmatrix}T(\alpha_0)\\N(\alpha_0)\end{bmatrix}$$

With initial tangent and normal vectors $T(\alpha_0)$ and $N(\alpha_0)$.

We can use Picard iterations to reconstruct curves from *non-analytic* equi-affine curvature. In the below example we use bump functions, which are smooth, non-analytic functions given by [2]:

$$b_{r_1,r_2}(s) = \begin{cases} 0 & s \leq r_1 \\ \dfrac{\exp\left(-\frac{1}{s-r_1}\right)}{\exp\left(-\frac{1}{s-r_1}\right) + \exp\left(-\frac{2}{r_1+r_2-2s}\right)} & r_1 < s < \frac{r_1+r_2}{2} \\ 1 & s = \frac{r_1+r_2}{2} \\ \dfrac{\exp\left(-\frac{1}{r_2-s}\right)}{\exp\left(-\frac{2}{2s-r_1-r_2}\right) + \exp\left(-\frac{1}{r_2-s}\right)} & \frac{r_1+r_2}{2} < s < r_2 \\ 0 & s \geq r_2. \end{cases}$$

We stitch together shifted bump functions to define the curvature $\mu(\alpha)$ shown on the left with its reconstructed curve on the right.



Reconstruction from Euclidean Curvature

Given the Euclidean curvature $\kappa(s)$, parameterized by Euclidean arclength s, of a curve γ, with initial point (x_0, y_0), and initial angle of the tangent vector with respect to the x-axis θ_0, the formulas to reconstruct an arclength parameterization of γ are as follows:

$$\theta(s) = \int_0^s \kappa(\tau)d\tau + \theta_0$$

$$x(s) = \int_0^s \cos(\theta(\tau))d\tau + x_0$$

$$y(s) = \int_0^s \sin(\theta(\tau))d\tau + y_0,$$

where $\gamma(s) = (x(s), y(s))$.

Closeness of Curvature

Lemma: Let κ_1, κ_2 be Euclidean (or equi-affine) curvatures defined on $[a, b]$ parameterized with respect to arc length (or affine arc length). For all $\varepsilon > 0$, there exists $\delta > 0$, such that if $|\kappa_1(t) - \kappa_2(t)| < \delta$ for $t \in [a, b]$, then there exists $g \in SE(2)$ (or $SA(2)$) such that $|\gamma_1(t) - g\gamma_2(t)| < \varepsilon$ for $t \in [a, b]$, where γ_1, γ_2 are the corresponding curve parameterizations.

Example: adding bump functions to $\sin(s)$ to get curvatures $\kappa_n(s) = \sin(s) + \frac{\pi}{n}b_{0,2}$.

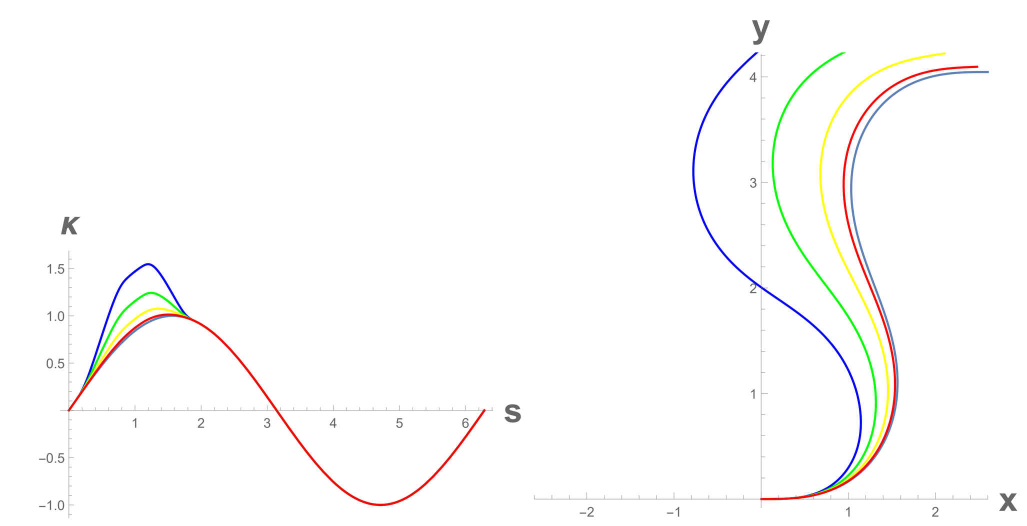


Fig. 3: Curvatures $\kappa_5(s)$, $\kappa_{10}(s)$, $\kappa_{25}(s)$, $\kappa_{100}(s)$, $\sin(s)$ and their reconstructed curves on the interval $[0, 2\pi]$

Caution! make sure that your intervals of reconstruction are fixed.

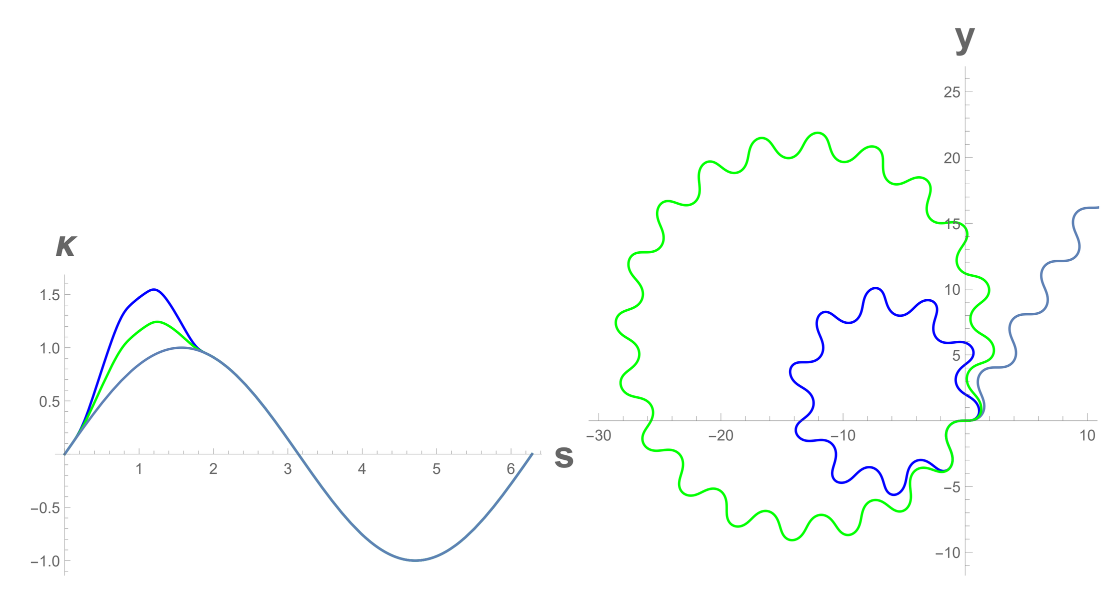


Fig. 4: Curvatures $\kappa_5(s)$, $\kappa_{10}(s)$, $\sin(s)$ and their respective reconstructed curves on various intervals

The reconstructed curves in Figure 4 are all closed [3], yet $\gamma_{\sin}$ is open. This appears to be a counterexample for our lemma, but upon further inspection, we discover that this phenomenon is due to not restricting the interval over which κ_n is defined. While it is true that each γ_{κ_n} is closed, the deviation from the original curve is small on a fixed interval. Figure 3 shows that the lemma does indeed hold, even for this seemingly problematic sequence.

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References

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