Rational and Replacement Invariants

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Group actions are ubiquitous in mathematics. They arise in diverse areas of applications, from mechanics to computer vision. A classical but central problem is to compute a complete set of invariants.

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Definitions

Algebraic Group $\mathcal{G}$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety $\quad G \subset \mathbb{K}[\lambda_1,\ldots,\lambda_l]$ = the ideal of $\mathcal{G}$

$$m : \mathcal{G}\times\mathcal{G}\to\mathcal{G},\ (\lambda,\mu)\mapsto \lambda\star\mu \qquad\text{and}\qquad i:\mathcal{G}\to\mathcal{G},\ \lambda\mapsto\lambda^{-1}$$

$$\lambda\star\lambda^{-1}=e \qquad e\star\lambda=\lambda\star e=\lambda$$

$$\lambda\star\mu\in\mathbb{K}[\lambda,\mu] \qquad\text{and}\qquad \lambda^{-1}\in\mathbb{K}[\lambda]$$

Rational Action on $\mathcal{Z}=\mathbb{K}^n$

$$\mathcal{G}\times\mathcal{Z}\to\mathcal{Z},\quad (\lambda,z)\mapsto\lambda\cdot z=\left(\frac{g_1(\lambda,z)}{h(\lambda,z)},\ldots,\frac{g_n(\lambda,z)}{h(\lambda,z)}\right) \qquad \lambda\cdot(\mu\cdot z)=(\lambda\star\mu)\cdot z,\quad e\cdot z=z$$

Orbit of $z\in\mathcal{Z}$: $\mathcal{O}_z=\{\lambda\cdot z\mid\lambda\in\mathcal{G}\}$

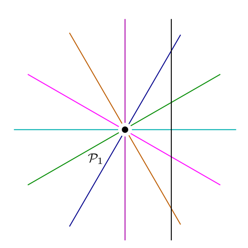
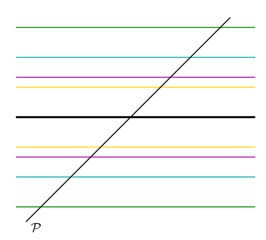
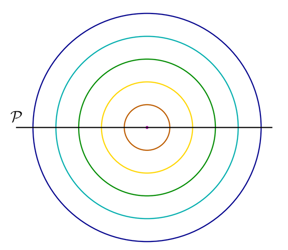
Field of Rational Invariants $\mathbb{K}(z)^G$

Rational invariant: $\frac{p}{q}\in\mathbb{K}(z)$ $\quad \frac{p(\lambda\cdot z)}{q(\lambda\cdot z)}=\frac{p(z)}{q(z)} \mod G$

Finiteness: $\mathbb{K}(z)^G=\mathbb{K}(r_1,\ldots,r_\kappa)\subset\mathbb{K}(z)$

Orbit separation: $z'\in\mathcal{O}_z \Leftrightarrow r_i(z)=r_i(z')$ for $z\in\mathcal{Z}\setminus\mathcal{W}$

Examples

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K}\times\{-1,1\}$	$SO(2)$
G	$(\lambda_1\lambda_2-1)$	(λ_2^2-1)	$(\lambda_1^2+\lambda_2^2-1)$
$\lambda\star\mu$	$(\lambda_1\mu_1,\ \lambda_2\mu_2)$	$(\lambda_1+\mu_1,\lambda_2\mu_2)$	$(\lambda_1\mu_1-\lambda_2\mu_2,\lambda_1\mu_2+\lambda_2\mu_1)$
e	$(1,1)$	$(0,1)$	$(1,0)$
λ^{-1}	$(\lambda_2,\ \lambda_1)$	$(-\lambda_1,\lambda_2)$	$(\lambda_1,-\lambda_2)$
	scaling	transla^o+reflec^o	rotation
$\lambda\cdot z$	$\begin{pmatrix}\lambda_1z_1\\ \lambda_1z_2\end{pmatrix}$	$\begin{pmatrix}z_1+\lambda_1\\ \lambda_2z_2\end{pmatrix}$	$\begin{pmatrix}\lambda_1&-\lambda_2\\ \lambda_2&\lambda_1\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$
$\mathcal{O}_z,\mathcal{P}$			
$\mathbb{K}(z)^G$	$\mathbb{K}\left(\frac{z_1}{z_2}\right)$	$\mathbb{K}\left(z_2^2\right)$	$\mathbb{K}\left(z_1^2+z_2^2\right)$
O^e	$Z_2-\frac{z_2}{z_1}Z_1$	$Z_2^2-z_2^2$	$Z_2^2+Z_1^2-(z_1^2+z_2^2)$
I^e	$Z_1-1,\ Z_2-\frac{z_2}{z_1}$	$Z_1-Z_2,\ Z_2^2-z_2^2$	$Z_2,\ Z_1^2-(z_1^2+z_2^2)$
ξ	$(1,\ \frac{z_2}{z_1})$	$(\pm z_2,\ \pm z_2)$	$(0,\ \pm\sqrt{z_1^2+z_2^2})$

Construction of Rational Invariants

Graph of the action

$$\mathcal{O}=\{(z,z')\in\mathcal{Z}\times\mathcal{Z}\mid\exists\lambda\in\mathcal{G}\text{ s.t. }z'=\lambda\cdot z\}$$

Its ideal $O=(G+(Z-\lambda\cdot z))\cap\mathbb{K}[z,Z]$

$$(Z-\lambda\cdot z)=(h\,Z_i-g_i\mid 1\le i\le n):h^\infty$$

O^e the extension of O to $\mathbb{K}(z)[Z]$.

Invariance

$(z,z')\in\mathcal{O}\Rightarrow(\lambda\cdot z,z')\in\mathcal{O} \qquad p(z,Z)\in O^e\Rightarrow p(\lambda\cdot z,Z)\in O^e$

$\boxed{\text{The reduced Gröbner basis of } O^e \text{ is contained in } \mathbb{K}(z)^G[Z].}$

PROOF:

$$Q\ni q(z,Z)=Z^A+\sum_a q_a(z)\,Z^a$$

$$O^e\ni q(\lambda\cdot z,Z)=Z^A+\sum_a q_a(\lambda\cdot z)\,Z^a$$

$$O^e\ni q(\lambda\cdot z,Z)-q(z,Z)=\sum_a(q_a(\lambda\cdot z)-q_a(z))\,Z^a=0$$

Generation

Q reduced Gröbner basis of O^e $\qquad \{r_1,\ldots,r_\kappa\}$ its coefficients

$$\boxed{\mathbb{K}(\mathbf{z})^{\mathbf{G}}=\mathbb{K}(\mathbf{r_1},\ldots,\mathbf{r_\kappa})}$$

Rewriting Algorithm $\xrightarrow{G}_Q$

IN: Q, $\ r=\frac{p}{q}\in\mathbb{K}(z)^G$

OUT: $R\in\mathbb{K}[y_1,\ldots,y_\kappa]$ s.t. $r=R(r_1,\ldots,r_\kappa)$

- $Q_y:=Q(r_i\leftarrow y_i)$; $\quad y_1,\ldots,y_\kappa$ new indeterminates
- $p(Z)\longrightarrow^*_{Q_y}\sum_\alpha a_\alpha(y)\,Z^\alpha \qquad q(Z)\longrightarrow^*_{Q_y}\sum_\alpha b_\alpha(y)\,Z^\alpha$
- $R:=\frac{a_\alpha}{b_\alpha}$ s.t. $b_\alpha(r)\neq 0$

KEY: $q(z)\,p(Z)\equiv p(z)q(Z) \mod O\Rightarrow q(z)a_\alpha(r)\,Z^\alpha=p(z)b_\alpha(r)\,Z^\alpha$

Replacement Invariants

Cross-section of degree d

a variety that intersects generic orbits in d simple points.

s = dimension of O^e = dimension of generic orbits

P defines a cross-section $\mathcal{P}$ of degree d:

- $P\subset\mathbb{K}[Z]$ prime ideal of codimension s
- $I^e=O^e+P$ radical and zero-dimensional
- $\dim_{\mathbb{K}(z)}\mathbb{K}(z)[Z]/I^e=d$

A generic affine space $P=(a_{i1}z_1+\ldots+a_{in}z_n-b_i,\ i=1..s)$ works

d is the degree of the generic orbits

Rational Invariants 2

Q a reduced Gröbner basis I^e $\qquad \{r_1,\ldots,r_\kappa\}$ its coefficients

$$\boxed{\begin{array}{c}\{\mathbf{r_1},\ldots,\mathbf{r_\kappa}\}\subset\mathbb{K}(\mathbf{z})^{\mathbf{G}}\\ \mathbf{r}\in\mathbb{K}(\mathbf{z})^{\mathbf{G}}\Rightarrow\mathbf{r}=\mathbf{R}(\mathbf{r_1},\ldots,\mathbf{r_\kappa})\text{ where }\mathbf{r}\xrightarrow{G}_{\mathbf{Q}}\mathbf{R}\end{array}}$$

Replacement Invariants $\xi=(\xi_1,\ldots,\xi_n)$

If $d=1$ then $Q=\{Z_i-r_i(z),i=1..n\}$ so that $r\xrightarrow{G}_Q r$ i.e.

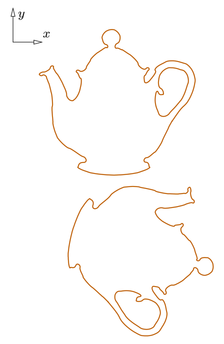
$$r(z_1,\ldots,z_n)=r(r_1,\ldots,r_n)$$

If $d>1$ then $I^G=I^e\cap\mathbb{K}(z)^G[Z]=(Q)$ has d zeros over $\overline{\mathbb{K}(z)}^G$.

$$\boxed{\xi\text{ a }\overline{\mathbb{K}(z)}^G\text{-zero of }I^G\Rightarrow r(z)=r(\xi),\quad\forall r\in\mathbb{K}(z)^G}$$

Equivalence of Curves $\Rightarrow$ Object Recognition

Isometries



$$\begin{pmatrix}X\\ Y_0\end{pmatrix}=\begin{pmatrix}\alpha&-\beta\\ \epsilon\beta&\epsilon\alpha\end{pmatrix}\begin{pmatrix}x\\ y_0\end{pmatrix}+\begin{pmatrix}a\\ b\end{pmatrix}$$

$$\alpha^2+\beta^2=1,\ \epsilon^2=1$$

$$\mathbb{K}(x,y_0)^G=\mathbb{K}$$

Local action on curves = Prolonged action on jets

$$Y_1=\frac{\beta+\alpha y_1}{\epsilon(\alpha-\beta y_0)},\quad Y_2=\frac{\epsilon\,y_2}{(\alpha-\beta y_0)^3},\quad Y_3=\frac{3\beta y_2^2+(\alpha-\beta y_1)y_3}{\epsilon(\alpha-\beta\,y_0)^5}$$

$$\mathbb{K}(x,y_0,y_1,y_2,y_3)^G=\mathbb{K}\left(\frac{y_2^2}{(1+y_1^2)^3},\frac{(y_3(1+y_1^2)-3y_1y_2^2)^2}{y_2^4}\right)$$

The curvature $\sigma=\frac{y_2^2}{(1+y_1^2)^{3/2}}$ and infinitesimal arclength $\mathrm{d}s^2=(1+y_1^2)\,\mathrm{d}x^2$ naturally arise in the replacement invariants

$$\xi^{(\pm)}=\left(0,0,0,\pm\sigma,\pm\frac{\mathrm{d}\sigma}{\mathrm{d}s}\right)$$

Two curves are locally equivalent by isometries if they imply the same relationship between σ and $\frac{\mathrm{d}\sigma}{\mathrm{d}s}$. This classical result generalizes to other equivalence problems [Fels & Olver 99].

Increasing the effect of bas-relief

The following action is taken in account in architecture to minimize the depth of bas relief without reducing their effect. We apply our algebraic construction to compute the differential invariant using the inductive strategy of [Kogan 03].

$$\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}=\begin{pmatrix}\lambda\,x-\mu\,y+\alpha\\ \mu\,x+\lambda\,y+\beta\\ \dfrac{a\,z+b}{c\,z+d}\end{pmatrix}\qquad\lambda^2+\mu^2=1$$

Consider $z=z(x,y)$

Second and third order differential invariants:

$$\xi_x=0,\ \xi_y=0,\ \xi_z=0,\ \xi_{z_x}=1,\ \xi_{z_y}=1,\ \xi_{z_{xy}}=0$$

$$\xi_{z_{xx}}=\eta_{z_{xx}}-\eta_{z_{xy}},\qquad\xi_{z_{yy}}=\eta_{z_{yy}}-\eta_{z_{xy}}$$

$$\xi_{z_{xxy}}=D_1(\eta_{z_{xy}})+2\eta_{z_{xx}}\eta_{z_{xy}}+\tfrac{1}{2}\eta_{z_{xy}}^2-2\eta_{z_{xx}}^2+2\eta_{z_{yy}}(\eta_{z_{xx}}-\eta_{z_{xy}})$$

where $\eta_{z_{xx}},\eta_{z_{xy}},\eta_{z_{yy}}$ are the second order differential invariants of the sub-action $c=0,d=1$

$$\eta_{z_{xx}}=\frac{2\left(z_x^2-z_y^2\right)z_{xy}-(z_y+z_x)^2\,z_{xx}-(z_y-z_x)^2\,z_{yy}}{\sqrt{2}\left(z_x^2+z_y^2\right)^{3/2}},$$

$$\eta_{z_{xy}}=\frac{\left(z_y^2-z_x^2\right)(z_{xx}-z_{yy})-4\,z_xz_yz_{xy}}{\sqrt{2}\,({z_x}^2+{z_y}^2)^{3/2}},$$

$$\eta_{z_{yy}}=\frac{2\left(z_y^2-z_x^2\right)z_{xy}-(z_y-z_x)^2\,z_{xx}-(z_x+z_y)^2\,z_{yy}}{\sqrt{2}\,({z_x}^2+{z_y}^2)^{3/2}}$$

and D_1 is an invariant derivation of this sub-action

$$D_1=\frac{z_x-z_y}{\sqrt{2}(z_x^2+z_y^2)^{1/2}}\frac{\partial}{\partial x}-\frac{z_x+z_y}{\sqrt{2}(z_x^2+z_y^2)^{1/2}}\frac{\partial}{\partial y}$$

The Bigger Project: Invariant Differential Systems

A Problem in Differential Elimination: Equations for s?

$$\mathcal{S}\begin{cases}s\,(\phi_{xx}+\phi_{yy})+s_x\,\phi_x+s_y\,\phi_y+\phi=0\\ s\,(\psi_{xx}+\psi_{yy})+s_x\,\psi_x+s_y\,\psi_y+\psi=0\\ \psi_x\,\phi_x+\psi_y\,\phi_y=0\end{cases}$$

Note: ranking on the derivatives breaks the symmetry

Idea : factor out the symmetry [Mansfield 01]

Tool : moving frame construction [Fels-Olver 99]

The symmetry group [Desolv]

$$X=\frac{\alpha}{\rho}x-\frac{\beta}{\rho}y+\frac{a}{\rho}\qquad Y=\frac{\alpha}{\rho}x+\frac{\beta}{\rho}y+\frac{b}{\rho}$$

$$S=\frac{s}{\rho^2\tau}\qquad\Phi=\frac{\phi}{\mu}\qquad\Psi=\frac{\psi}{\nu}$$

Fundamental invariants [Vessiot, Groebner]

$$\psi_1=\frac{s_y\psi_x-s_x\psi_y}{2\,s_1\,\psi},\qquad P=(x,y,s-1,\phi-1,\psi-1,s_x)$$

$$\psi_2=\frac{s_x\psi_x+s_y\psi_y}{2\,s_1\,\psi},\quad s_1^2=\frac{s_x^2+s_y^2}{4\,s},$$

$$\phi_1=\frac{s_y\phi_x-s_x\phi_y}{2\,s_1\,\phi},\quad s_2=\frac{s_{xy}(s_y^2-s_x^2)+s_xs_y(s_{xx}-s_{yy})}{8\,s\,s_1^3}$$

$$\phi_2=\frac{s_x\phi_x+s_y\phi_y}{2\,s_1\,\phi},\quad s_3=\frac{s_x^2s_{yy}+s_y^2s_{xx}-2\,s_xs_ys_{xy}-8\,s\,s_1^4}{8\,s\,s_1^3}$$

Invariant derivations

$$\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=\frac{\pm\sqrt{s\,(s_y^2+s_x^2)}}{s_x^2-s_y^2}\begin{pmatrix}-s_y&s_x\\ s_x&-s_y\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}$$

Algebra of Differential Invariants

Non-commutation of the derivations: [ivb]

$$\delta_1\delta_2-\delta_2\delta_1=s_3\,\delta_1+s_2\,\delta_2$$

Syzygies of the fundamental invariants: [aida]

$$\mathcal{Z}\begin{cases}\delta_1(s_1)=s_1s_2\\ \delta_1(s_2)-\delta_2(s_3)=s_3^2+s_2^2+s_1\,(s_2+s_3)\\ \delta_1(\phi_2)-\delta_2(\phi_1)=\phi_1\,s_3+\phi_2\,s_2,\\ \delta_1(\psi_2)-\delta_2(\psi_1)=\psi_1\,s_3+\psi_2\,s_2.\end{cases}$$

Differential Rewriting of $\mathcal{S}$ [aida]

in terms of $\mathcal{Y}=\{s_1,s_2,s_3,\phi_1,\phi_2,\psi_1,\psi_2\}$ and $\Delta=\{\delta_1,\delta_2\}$:

$$\mathcal{S}^{\mathcal{G}}\begin{cases}\delta_1(\phi_1)+\delta_2(\phi_2)+\phi_1^2+\phi_2^2-s_2\phi_1+(2\,s_1+s_3)\phi_2+1=0,\\ \delta_1(\psi_1)+\delta_2(\psi_2)+\psi_1^2+\psi_2^2-s_2\psi_1+(2\,s_1+s_3)\psi_2+1=0,\\ \phi_1\,\psi_1+\phi_2\,\psi_2=0.\end{cases}$$

Invariant Differential Elimination [diffalg]

Treat $\mathcal{Z}\cup\mathcal{S}^{\mathcal{G}}$ in the differential polynomial ring with non trivial commutation rules [Hubert 05].

Tips & Techniques

The cross-section is chosen so as s_1,s_2,s_3 and δ_1,δ_2 depend only on s and its derivatives.

To simplify the commutation rules we chose

$s_1=\xi_{s_y}/2,\ s_2=\xi_{s_{xy}}/\xi_{s_y},\ s_3=\xi_{s_{xx}}/\xi_{s_y}-s_1,\ \psi_1=\xi_{\psi_x},\ \psi_2=\xi_{\psi_y},\ \phi_1=\xi_{\phi_x},\ \phi_2=\xi_{\phi_y}$