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Hyperbolic conservation laws with prescribed eigenfields.

Irina Kogan

North Carolina State University

joint work with

Michael Benfield , NC Sate

and

Kris Jenssen, Penn State

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System of conservation laws

$$u_t + f(u)_x = 0\,. \qquad (1a)$$

- n equations on n unknown functions $u(x,t) \in \Omega \subset \mathbb{R}^n$.
- one space-variable $x \in \mathbb{R}$; one time-variable: $t \in \mathbb{R}$.
- $f(u) : \Omega \to \mathbb{R}^n$ smooth flux.

$$u_t + [D_u f]\, u_x = 0 \qquad (1b)$$

- (1) is called hyperbolic on Ω if $\forall \bar{u} \in \Omega$ the Jacobian matrix $[D_u f]$ is diagonalizable over $\mathbb{R}$.

  $$\Updownarrow$$

  eigenvector fields $R_1(u), \ldots, R_n(u)$ of $[D_u f]$ are independent at $\forall \bar{u} \in \Omega$ – they comprise an eigenframe.

- $\lambda^1(u), \ldots, \lambda^n(u)$ are eigenfunctions of $D_u f$. If distinct $\forall u \in \Omega$, then (1) is called strictly hyperbolic.

Problem: Given a set independent vector fields $\mathcal{R} = \{R_1, \ldots, R_m\}$, $1 \leq m \leq n$ on open $\Omega \subset \mathbb{R}^n$, find all maps $f : \Omega \to \mathbb{R}^n$ (fluxes), whose Jacobian matrix $[D_u f]$ has $\mathcal{R}$ as a prescribed (partial) set of eigenvector-fields.

Motivation:

- Construct conservations laws with prescribed rarefaction curves and analyze how the geometry of these curves determines behavior of the solutions of conservative these systems.
- Interesting geometric problem on its own.
- Leads to interesting overdetermined systems of PDE's.

Examples: full frames in $\mathbb{R}^3$ (coordinates (u, v, w))

(1) $R_1 = [0, 1, u]^T$, $R_2 = [w, 0, 1]^T$, $R_3 = [u, 0, -w]^T$
4-dimensional space of trivial fluxes:

$$f = a\,[u, v, w]^T + [b_1, b_2, b_3]^T \text{ where } a,\ b_1, b_2, b_3 \in \mathbb{R}$$

$$Df = a\,I \quad \Longrightarrow \quad \lambda^1 = \lambda^2 = \lambda^3 = a$$

(2) $R_1 = [v,\ u,\ 1]^T, \quad R_2 = [-v,\ u,\ 0]^T, \quad R_3 = [0,\ 0,\ 1]^T$
5-dimensional vector space of fluxes

$$f = c\left[v^3,\ u^3,\ \frac{3}{4}(u^2 + v^2)\right]^T + \text{a trivial flux}, \quad c \in \mathbb{R}$$

$$\lambda^1 = 3\,c\,u\,v + a, \quad \lambda^2 = -3\,c\,u\,v + a, \quad \lambda^3 = a.$$

(3) $R_1 = [1,\ 0,\ 0]^T, \quad R_2 = [0,\ 1,\ 0]^T, \quad R_3 = [0,\ 0,\ 1]^T$

$$f = \left[\phi^1(u),\ \phi^2(v),\ \phi^3(w)\right]^T, \quad \phi^i \colon \mathbb{R} \to \mathbb{R} \text{ arbitrary}$$

$$\lambda^1 = \left(\phi^1\right)'(u), \quad \lambda^2 = \left(\phi^2\right)'(v), \quad \lambda^3 = \left(\phi^3\right)'(w).$$

What if we prescribe incomplete (partial) eigenframe?

(1) $R_1 = [0, 1, u]^T,\ R_2 = [w, 0, 1]^T,\ R_3 = [u, 0, -w]^T$ only trivial fluxes.

(1a) $\boxed{R_1 = [0, 1, u]^T,\ R_2 = [w, 0, 1]^T}$ again only trivial fluxes!

(1b) $\boxed{R_1 = [0, 1, u]^T}$, $\boxed{R_3 = [u, 0, -w]^T.}$

$$f = c_1 \begin{bmatrix} \ln(u) \\ 0 \\ \frac{1}{2}\left(\frac{w}{u} - v\right) \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{3} u^3 \\ u\,w \\ w\,u^2 \end{bmatrix} + a \begin{bmatrix} u \\ v \\ v \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$F^1 = c_1 \ln(u) - \frac{1}{3} c_2\, u^3, \quad F^2 = c_2\, u\, w, \quad F^3 = \frac{1}{2} c_1 \left(\frac{w}{u} - v\right) + c_2\, u^2.$$

$$\lambda^1 = c_2\, u^2, \quad \lambda^3 = c_1 \frac{1}{u} - c_2\, u^2$$

(1c) $\boxed{R_2 = [w, 0, 1]^T,\ R_3 = [u, 0, -w]^T.}$

∞-dimensional family of fluxes, but no strictly hyperbolic among them!

What about coordinate frame example?

(3) $R_1 = [1,\ 0,\ 0]^T, \quad R_2 = [0,\ 1,\ 0]^T, \quad R_3 = [0,\ 0,\ 1]^T$

$$f = \left[\phi^1(u),\ \phi^2(v),\ \phi^3(w)\right]^T, \quad \phi^i\colon \mathbb{R} \to \mathbb{R} \text{ arbitrary}$$

$$\lambda^1 = (\phi^1)'(u), \quad \lambda^2 = (\phi^2)'(v), \quad \lambda^3 = (\phi^3)'(w).$$

(3a) $\boxed{R_1 = [1,\ 0,\ 0]^T, \quad R_2 = [0,\ 1,\ 0]^T.}$

$$f = \left[\phi^1(u,w),\ \phi^2(v,w),\ \phi^3(w)\right]^T, \quad \phi^1, \phi^2\colon \mathbb{R}^2 \to \mathbb{R}; \quad \phi^3\colon \mathbb{R}^2 \to \mathbb{R}$$

$$\lambda^1 = \frac{\partial \phi^1}{\partial u}, \quad \lambda^2 = \frac{\partial \phi^2}{\partial v}.$$

(3b) $\boxed{R_1 = [1,\ 0,\ 0]^T.}$

$$f = \left[\phi^1(u,v,w),\ \phi^2(v,w),\ \phi^3(v,w)\right]^T \quad \phi^1\colon \mathbb{R}^3 \to \mathbb{R}; \quad \phi^2, \phi^3\colon \mathbb{R}^2 \to \mathbb{R}$$

$$\lambda^1 = \frac{\partial \phi^1}{\partial u}.$$

How did we find f in the above examples?

- Given a set of independent vector-fields $\mathcal{R} = \{R_1, \ldots, R_m\}$ on $\Omega \subset \mathbb{R}^n$, set up an overdetermined (for $n > 2$) system of $m\,n$ 1st order PDE's on $n + m$ unknown functions $f = [F^1, \ldots, F^n] \colon \Omega \to \mathbb{R}^n$ and $\lambda^i \colon \Omega \to \mathbb{R}$, $i = 1, \ldots, m$.

  $$[D_u f]\, R_i(u) = \lambda^i(u)\, R_i(u), \quad i = 1, \ldots, m$$

  $\boxed{\mathcal{F}(\mathcal{R})\text{-system}}$

  where $[D_u f] = \left[\frac{\partial F^i}{\partial u^j}\right]_{i,j=1,\ldots,n}$ is the Jacobian matrix.

  (Although unknown functions $\lambda^i, \ldots, \lambda^m$ are not differentiated, they are not free parameters, but must, for $n > 1$ satisfy some conditions for $\mathcal{F}(\mathcal{R})$-system to have a solution.)

- Either solve by hand or employ a computer solver (e.g. Maple, "pdsolve", and hope that it produces a complete and readable solutions set of $\mathcal{F}(\mathcal{R})$-system).

- Can we trust these computations?!!

Can we <u>predict</u> the <u>"size"</u> and the <u>structure</u> of the solution set of $\mathcal{F}(\mathcal{R})$-system <u>from the geometric properties</u> of the set $\mathcal{R} = \{R_1, \ldots, R_m\}$?

Yes, by using integrability theorems: smooth Frobenius and Darboux theorems (and their generalizations), and as the last resort analytic Cartan-Kähler theorem.

Geometry of vector fields

vector fields $\longleftrightarrow$ derivations:

$$S(u) = [S^1(u), \dots, S^n(u)] \quad \longleftrightarrow \quad \mathbf{s} = S^1(u)\frac{\partial}{\partial u^1} + \cdots + S^n(u)\frac{\partial}{\partial u^n}.$$

$$\phi\colon \Omega \to \mathbb{R}, \quad \mathbf{s}\colon \phi \to \mathbf{s}(\phi) = S \cdot \mathsf{grad}\, \phi.$$

Notation: $C^\infty(\Omega)$ - the set of smooth functions on Ω;
$\mathcal{X}^\infty(\Omega)$ - the set of smooth vector-fields on Ω.

Covariant derivative: For $\mathbf{s}, \mathbf{r} \in \mathcal{X}(\Omega)$ define

$$\nabla_{\mathbf{s}}\mathbf{r} := \mathbf{s}(R) \in \mathcal{X}(\Omega)$$

where R is a component vector of $\mathbf{r}$ in u-coordinates, and $\mathbf{s}$ is applied to each component.

Lie bracket: $[\mathbf{r}, \mathbf{s}](\phi) := \mathbf{r}(\mathbf{s}(\phi)) - \mathbf{s}(\mathbf{r}(\phi))$

Theorem: $[\mathbf{r}, \mathbf{s}] = \nabla_{\mathbf{r}}\mathbf{s} - \nabla_{\mathbf{s}}\mathbf{r} \in \mathcal{X}(\Omega)$

In more intrinsic geometric language:

We defined flat, symmetric connection ∇ on Ω, with u being affine coordinates:

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0, \quad \forall i, j = 1, \ldots, n.$$

For all $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \mathcal{X}(\Omega)$,

$$\begin{aligned} \nabla_{\mathbf{r}}\mathbf{s} - \nabla_{\mathbf{s}}\mathbf{r} &= [\mathbf{r}, \mathbf{s}] && \text{Symmetry,} \\ \nabla_{\mathbf{r}}\nabla_{\mathbf{s}}\,\mathbf{t} - \nabla_{\mathbf{s}}\nabla_{\mathbf{r}}\mathbf{t} &= \nabla_{[\mathbf{r},\mathbf{s}]}\mathbf{t} && \text{Flatness.} \end{aligned}$$

Partial frames, involutivity, richness

Definitions:

- A set of smooth vector fields $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$, where $m \leq n$, is called a partial frame on open $\Omega \subset \mathbb{R}^n$ if at each $\bar{u} \in \Omega$ vectors $\mathbf{r}_1|_{\bar{u}}, \ldots, \mathbf{r}_m|_{\bar{u}}$ are linearly independent. If $m = n$, then $\mathcal{R}$ is a frame.

- $\mathcal{R}$ is in involution if $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty} \mathcal{R}$ for all $1 \leq i, j \leq m$.

- $\mathcal{R}$ is rich if $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty}\{\mathbf{r}_i, \mathbf{r}_j\}$ (pairwise in involution).

Darboux Integrability Theorem [Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910)]

Given:

1. subsets $\alpha(i) \subset \{1, \dots, n\}$ for each $i = 1, \dots, p$.
2. $\Omega \subset \mathcal{R}^n$ and $\Theta \subset \mathbb{R}^p$ open subsets
3. $h^i_j(u^1, \dots, u^n,\ \phi^1, \dots, \phi^p), \quad i = 1, \dots, p,\ j \in \alpha(i)$ smooth functions on $\Omega \times \Theta \to \mathbb{R}$, with certain combinatorial restrictions on which ϕ's each of the h^i_j may depend so that (2) become algebraic.

Consider a system of PDE's on $(\phi^1, \dots \phi^p) : \Omega \to \Theta$:

$$\boxed{\frac{\partial \phi^i}{\partial u^j} = h^i_j(u, \phi(u))}, \quad i = 1, \dots, p;\ j \in \alpha(i). \tag{1}$$

If integrability conditions

$$\frac{\partial}{\partial u^k}\left(\frac{\partial}{\partial u^j}(\phi^i)\right) - \frac{\partial}{\partial u^j}\left(\frac{\partial}{\partial u^k}(\phi^i)\right) = 0 \text{ for all } j, k \in \alpha(i) \tag{2}$$

are identically satisfied on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\frac{\partial}{\partial u^j}(\phi^i)$ for all $i = 1, \dots, p,\ j \in \alpha(i)$ as prescribed by system (1)

Then $\exists!$ smooth local solution of (1) around $\bar{u}$, for any smooth initial data for ϕ^i prescribed along submanifold $\Xi_i = \{u^j = \bar{u}^j,\ j \in \alpha_i\} \subset \mathbb{R}^n$ of dimension $n - |\alpha_i|$.

Frobenius Theorem: PDE version: suff. cond. [Deahna (1840)]; nec. cond. [Clebsch (1860)]; diff. form version: [Frobenius (1877)]: vectorfield formulation: (all equivalent)

Generalized PDE version [M. Benfield (2016)]:

Given:

1. $\mathcal{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ – a partial frame in involution on open $\Omega \subset \mathcal{R}^n$.
2. $\Theta \subset \mathbb{R}^p$ is open
3. $h^i_j(u, \phi), \quad i = 1, \dots, p,\ j = 1, \dots, m$ smooth functions on $\Omega \times \Theta \to \mathbb{R}$.

Consider a system of PDE's on $(\phi^1, \dots \phi^p)$: $\Omega \to \Theta$:

$$\boxed{\mathbf{r}_j(\phi^i(u)) = h^i_j(u, \phi(u))}, \quad i = 1, \dots, p;\ j = 1, \dots, m. \tag{3}$$

If integrability conditions

$$\mathbf{r}_k\left(\mathbf{r}_j(\phi^i)\right) - \mathbf{r}_j\left(\mathbf{r}_k(\phi^i)\right) = \sum_{l=1}^{m} c^l_{jk}\mathbf{r}_l(\phi) \quad i = 1, \dots, p;\ j, k = 1, \dots, m \tag{4}$$

are identically satisfied on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\mathbf{r}_j(\phi^i)$ for all $i = 1, \dots, p,\ j = 1, \dots, m$ as prescribed by system (3)

Then $\exists!$ smooth local solution of (3), for any smooth initial data prescribed along any embedded submanifold $\Xi \subset \Omega$ of dimension $n - m$ transversal to $\mathcal{R}$.

Coordinate-free definition of the Jacobian map:

Definition: The Jacobian of a vector field $\mathbf{f}$ on open $\Omega \subset \mathbb{R}^n$, relative to a flat, symmetric connection on Ω connection ∇ is a map

$$J\mathbf{f}: \mathcal{X}(\Omega) \to \mathcal{X}(\Omega) \text{ defined by} J\mathbf{f}(\mathbf{r}) = \nabla_{\mathbf{r}}\mathbf{f}$$

If $\mathbf{f} = F^1 \frac{\partial}{\partial u^1} + \cdots + F^n \frac{\partial}{\partial u^n}$ and $\mathbf{r} = R^1 \frac{\partial}{\partial u^1} + \cdots + R^n \frac{\partial}{\partial u^n}$, where $u^1, \ldots, u^n$ are affine coordinates $\left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0 \right)$ then

$$J\mathbf{f}(\mathbf{r}) = [D_u F] R,$$

where $F = [F^1, \ldots, F^n]^T$ and $R = [R_1, \ldots, R^n]^T$.

Definition: $\mathbf{f}$ is called hyperbolic on Ω if eigenvector-fields of $J\mathbf{f}$ form a frame on Ω. (This implies that all eignefunctions of $J\mathbf{f}$ are real)

$\mathbf{f}$ is called strictly hyperbolic if, in addition, at every point of Ω all n eignefunctions of $J\mathbf{f}$ have distinct values.

Jacobian problem:

Given a partial frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ on open $\Omega \subset \mathbb{R}^n$ ($n \geq m$), and a fixed point $\bar{u} \in \Omega$, describe the set of smooth vector fields

$$\mathcal{F}(\mathcal{R}) = \{\mathbf{f} \in \mathcal{X}(\Omega') \,|\, \bar{u} \in \Omega' \subset \Omega\}$$

s. t. there $\exists$ smooth functions $\lambda^i \colon \Omega' \to \mathbb{R}$ for which

$$J\mathbf{f}(\mathbf{r}_i) := \nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \, \mathbf{r}_i, \qquad \text{for } i = 1, \ldots, m,$$

where ∇ is a flat, symmetric connection on Ω.

Elements of $\mathcal{F}(\mathcal{R})$ will be called fluxes.

- $\mathcal{F}(\mathcal{R})$ is, possibly ∞-dimensional, $\mathbb{R}$-vector space.
- scaling invariance: if $\tilde{\mathcal{R}} = \{\phi^1 \, \mathbf{r}_1, \ldots, \phi^m \, \mathbf{r}_m\}$, where $\phi^i \colon \Omega \to \mathbb{R}$ are nowhere zero, then $\mathcal{F}(\mathcal{R}) = \mathcal{F}(\tilde{\mathcal{R}})$.
- $\forall \mathcal{R}$, the set $\mathcal{F}(\mathcal{R})$ contains a trivial fluxes:

$$(a\, u^1 + b^1)\frac{\partial}{\partial u^1} + \cdots + (a\, u^n + b^n)\frac{\partial}{\partial u^n}, \text{ for all } a, b^1, \ldots, b^n \in \mathbb{R}.$$

Jacobian problem for $\boxed{\text{rich}}$ (partial) frames $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$:

Recall:

- rich means that $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty}\{\mathbf{r}_i, \mathbf{r}_j\}$ $1 \le i, j \le m$.
- $\mathbf{f} \in \mathcal{F}(\mathcal{R})$ if $\exists \lambda^i \colon \Omega \to \mathcal{R}$ such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m.$$

Theorem: If $\mathcal{R}$ is rich then $\mathcal{F}(\mathcal{R})$ $\boxed{\text{contains strictly hyperbolic fluxes iff}}$

$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty}\{\mathbf{r}_i, \mathbf{r}_j\} \text{ for all } 1 \le i, j \le m. \qquad (*)$$

Under (*), $\mathcal{F}(\mathcal{R})$ depends on m arbitrary functions of $n - m + 1$ (the degree of freedom of prescribing λ's) and n functions of $n - m$ variables (the degree of freedom for prescribing $\mathbf{f}$ for given λ's)

Jacobian problem for non-involutive partial frames simplest case: $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ in $\mathbb{R}^3$.

Recall:

- non-involutive means that $[\mathbf{r}_1, \mathbf{r}_2] \notin \mathrm{span}_{C^\infty}(\mathcal{R}) = \mathrm{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\}$.
- $\mathbf{f} \in \mathcal{F}(\mathcal{R})$ if $\exists \lambda^1, \lambda^2 \colon \Omega \to \mathcal{R}$ such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \, \mathbf{r}_i, \quad \text{for } i = 1, 2.$$

Theorem: In this case, if $\mathcal{F}(\mathcal{R})$ $\boxed{\text{contains strictly hyperbolic fluxes}}$, then

$$\nabla_{\mathbf{r}_1}\mathbf{r}_2 \notin \mathrm{span}_{C^\infty}\{\mathbf{r}_i, \mathbf{r}_j\} \text{ and } \nabla_{\mathbf{r}_2}\mathbf{r}_1 \notin \mathrm{span}_{C^\infty}\{\mathbf{r}_i, \mathbf{r}_j\} \qquad (**)$$

Under (**), $\boxed{4 \le \dim(\mathcal{F}(\mathcal{R})) \le 8}$

- we have examples in all dimensions: $4, \dots, 8$ and with strictly hyperbolic fluxes when $\dim \mathcal{F}(\mathcal{R}) > 4$.
- $\dim \mathcal{F}(\mathcal{R}) = 4$ iff $\mathcal{F}(\mathcal{R})$ has only trivial fluxes.

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Thank you!