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Generalization of an integrability theorem of Darboux.

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joint work with

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Integrability theorems for PDEs: – theorems about local existence and the "size" of the solution set for an overdetermined system of PDEs. **The "size"** of the solution set is the number of arbitrary functions and constants the general solution depends on. Equivalently, it is the type of data that can be prescribed to guarantee the uniqueness of the solution.

- Cartan-Kähler theorem – the most general and powerful, but requires analyticity of the equations and the data.

- Darboux [Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910)] – quite specialized, but requires only C^1-regularity of the equations and the data.

- PDE version of the Frobenius integrability theorem is a particular case of the Darboux theorem.

We formulate and prove a generalization of the Darboux theorem.

Motivation: Geometric study of systems of hyperbolic conservation laws $U_t + F(U)_x = 0$.

A sub-problem (the Jacobian Problem):

Given a local frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ on $\Omega \subset \mathbb{R}^n$, find all maps $F\colon \Omega \to \mathbb{R}^n$ such that $\mathcal{R}$ is the set of eigenvectors of the Jacobian matrix DF.

This, in turn, leads to a PDE system of the type:

$$\mathbf{r}_i(u_\alpha)\Big|_x = f_i^\alpha(x, u(x)) \qquad \text{for } \alpha \in \{1, \ldots, m\} \text{ and } i \in I_\alpha \subseteq \{1, \ldots, n\}.$$

($u_\alpha\colon \mathbb{R}^n \to \mathbb{R}$, I_α may vary with α.)

Darboux Théorème III

Chapitre I, Livre III, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910).

Consider a system of PDEs:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

where

- $x = (x_1, \ldots, x_n)$ are independent variables;

- $u = (u_1, \ldots, u_m)$ are unknown functions;

- $I_\alpha \subseteq \{1, \ldots, n\}$ determines the set of partial derivatives $\partial_{x_i} u_\alpha$ prescribed by the system for the unknown function u^α.

- $f_i^\alpha(x, u)$ are given C^1-functions on $\mathbb{R}^n \times \mathbb{R}^m$.

Example of Darboux-type system:

- system:

$$\begin{aligned} u_x &= f(x, y, u, v) \\ v_x &= \phi(x, y, u, v) \\ v_y &= \psi(x, y, u, v) \end{aligned}$$

- two unknown functions u and v of (x, y).
- $I_u = \{1\}$ and $I_v = \{1, 2\}$.
- f, ϕ, ψ are given C^1-functions of (x, y, u, v).

Returning to the Darboux theorem

the system:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\Big(x, u(x)\Big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

with the data prescribed near a point $\bar{x} \in \mathbb{R}^n$ by:

$$u_\alpha|_{\Xi_\alpha} = g_\alpha, \quad \alpha = 1, \ldots, m,$$

where

- $\Xi_\alpha = \{x \mid x_i = \bar{x}_i, \text{ for all } i \in I_\alpha\}$
- g_α is an arbitrary C^1-functions on Ξ_α

Under appropriate integrability conditions

has a unique local C^1-solution near $\bar{x}$.

Example (data): for the system

$$\begin{aligned} u_x &= f(x, y, u, v) \\ v_x &= \phi(x, y, u, v) \\ v_y &= \psi(x, y, u, v) \end{aligned}$$

we prescribe data near $(\bar{x}, \bar{y})$:

$$u(\bar{x}, y) = g_1(y), \quad v(\bar{x}, \bar{y}) = g_2,$$

where $g(y)$ is an arbitrary C^1-function of one variable, g_2 is a constant

Example (integrability conditions):

$$
\begin{aligned}
u_x &= f(x, y, u, v) \\
v_x &= \phi(x, y, u, v) \\
v_y &= \psi(x, y, u, v)
\end{aligned}
$$

Equality of partials $v_{xy} = v_{yx}$ imposes a condition on f, ϕ, ψ:

$$\phi_y + \phi_u\, u_y + \phi_v\, v_y = \psi_x + \psi_u\, u_x + \psi_v\, v_x$$

$\Downarrow$ substitute $u_x, u_y, v_x,$ and v_y from the system ...

Example (integrability conditions):

$$\begin{aligned} u_x &= f(x,y,u,v) \\ v_x &= \phi(x,y,v) \qquad \boxed{\phi_u = 0} \\ v_y &= \psi(x,y,u,v) \end{aligned}$$

Equality of partials $v_{xy} = v_{yx}$ imposes a condition on f, ϕ, ψ:

$$\phi_y + \phi_u\, u_y + \phi_v\, v_y = \psi_x + \psi_u\, u_x + \psi_v\, v_x$$

$\Downarrow$ substitute $u_x, u_y, v_x,$ and v_y from the system :

$$\boxed{\phi_y + \phi_v\, \psi = \psi_x + \psi_u\, f + \psi_v\, \phi}$$

The Darboux theorem implies that a system:

$$\begin{aligned} u_x &= f(x,y,u,v) \\ v_x &= \phi(x,y,v) \\ v_y &= \psi(x,y,u,v) \end{aligned} \qquad \boxed{\phi_u = 0}$$

with the data

$$u(\bar{x}, y) = g_1(y), \quad v(\bar{x}, \bar{y}) = g_2,$$

where $g(y)$ is an arbitrary C^1-function of one variable, g_2 is a constant and f, ψ, ϕ are C^1-functions such that the equality

$$\boxed{\phi_y + \phi_v \psi = \psi_x + \psi_u f + \psi_v \phi}$$

is identically satisfied in a neighborhood of a point $(\bar{x}, \bar{y}, g_1(\bar{y}), g_2) \in \mathbb{R}^2 \times \mathbb{R}^3$,

has a unique C^1-solution near $(\bar{x}, \bar{y})$.

The Darboux theorem (Théorème III)

A system:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\Big(x, u(x)\Big), \quad i \in I_\alpha \subseteq \{1, \dots, n\},$$

with the data prescribed near a point $\bar{x} \in \mathbb{R}^n$ by:

$$u_\alpha|_{\Xi_\alpha} = g_\alpha, \quad \alpha = 1, \dots, m, \quad \text{where}$$

- $\Xi_\alpha = \{x \,|\, x_i = \bar{x}_i, \text{ for all } i \in I_\alpha\}$
- g_α is an arbitrary C^1-function on Ξ_α
- $f_i^\alpha(x, u)$ are C^1-functions satisfying near $(\bar{x}, g(\bar{x})) \in \mathbb{R}^n \times \mathbb{R}^m$:

$\boxed{\forall \alpha \text{ and } \forall i, j \in I_\alpha, \text{ such that } i \neq j:}$

1. $\boxed{\forall \beta \in \{1, \dots, m\}, \text{ if } i \notin I_\beta \text{ then } \partial_{u_\beta} f_j^\alpha \equiv 0}$

2. $$\boxed{\partial_{x_i} f_j^\alpha + \sum_{\beta : i \in I_\beta} \left(\partial_{u_\beta} f_j^\alpha\right) f_i^\beta \equiv \partial_{x_j} f_i^\alpha + \sum_{\beta : j \in I_\beta} \partial_{u_\beta} f_i^\alpha \, f_j^\beta}$$

has a unique local C^1-solution near $\bar{x}$.

Particular cases

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

- if, for all α, $|I_\alpha| = 1$ then the system is determined.
  (Darboux's Théorème I)

- if, for all α, $|I_\alpha| = n$ then the system is Frobenius
  (Darboux's Théorème II)

Outline of Darboux's proof

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \dots, n\},$$

- Darboux's Théorème I ($|I_\alpha| = 1$ for all α) is proved via Picard iterations.
- Darboux's Théorème III ($|I_\alpha|$ is arbitrary)

  Darboux wrote out a proof only for $n = 2$ and $n = 3$:

  "Pour établier cette importante proposition, sans employer un trop grand luxe de notations, nous nous bornerons au cas de deux et de trois variables indépendantes, qui suffira d'ailleurs pour les applications que nous avons en vue".

  - for $n = 2$ the proof uses Théorème I.
  - for $n = 3$ Darboux identifies sub-systems that can be treated by Théorème I or by $n = 2$ case. These sub-systems are solved in a "right" order so that the solution of one sub-system provides initial data to the next.

This suggests a proof by induction.

Extending Darboux's argument to an inductive proof for an arbitrary number of independent variables turned out to be non-trivial:

Benfield, Jenssen, and IK, ”On two theorems of Darboux” (2017) preprint, 27 pp

`http://www.math.ncsu.edu/~iakogan/papersPDF/BJK-dar.pdf`

We realized that a direct proof of a more general version of the theorem can be given.

Generalization of Théorème III

Our theorem generalizes Darboux's in two ways:

(i) Instead of partial derivatives, the directional derivatives of the unknown functions are prescribed along C^1-vector fields comprising a local frame $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ near $\bar{x}$:

$$\mathbf{r}_i(u_\alpha)\Big|_x = f_i^\alpha(x, u(x)) \qquad \text{for each } i \in I_\alpha \subseteq \{1, \ldots, n\}, .$$

(I_α may vary with α.)

(ii) The prescribed data g_α for unknown u_α may be given along an arbitrary $(n - |I_\alpha|)$-dimensional manifold through the point $\bar{x}$, transversal to the vector fields $\{\mathbf{r}_i \,|\, i \in I_\alpha\}$.

Integrability conditions are imposed:

by the requirement that the derivatives, prescribed by the system, are consistent with the structure equations of the frame:

$$[\mathbf{r}_i, \mathbf{r}_j] = \sum_{k=1}^{n} c_{ij}^k \mathbf{r}_k.$$

In other words, we substitute the derivatives $\mathbf{r}_j(u_\alpha)$ prescribed by the system into

$$\mathbf{r}_i\Big(\mathbf{r}_j(u_\alpha)\Big) - \mathbf{r}_j\Big(\mathbf{r}_i(u_\alpha)\Big) = \sum_{k=1}^{n} c_{ij}^k \mathbf{r}_k(u_\alpha) \qquad (*)$$

and require that:

1. No unprescribed derivatives of u_α are present in (*).
2. Equality (*) holds as an identity near $(\bar{x}, g(\bar{x})) \in \mathbb{R}^n \times \mathbb{R}^m$.

Integrability conditions in details:

$$\mathbf{r}_i\Big(\mathbf{r}_j(u_\alpha)\Big) - \mathbf{r}_j\Big(\mathbf{r}_i(u_\alpha)\Big) = \sum_{k=1}^{n} c_{ij}^{k}\,\mathbf{r}_k(u_\alpha) \qquad (*)$$

$\forall \alpha$ and $\forall i, j \in I_\alpha$, such that $i \neq j$:

1. No unprescribed derivatives of u_α are present in (*):

   - $\forall \beta \in \{1, \ldots, m\}$, if $i \notin I_\beta$ then $\partial_{u_\beta} f_j^\alpha \equiv 0$

   - if $k \notin I_\alpha$ then $c_{ij}^k \equiv 0$

2. Equality (*) holds as an identity near $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$D_x f_j^\alpha(x,u) \cdot \mathbf{r}_i\Big|_x + \sum_{\beta : i \in I_\beta} \partial_{u_\beta} f_j^\alpha(x,u) f_i^\beta(x,u)$$

$$-D_x f_i^\alpha(x,u) \cdot \mathbf{r}_j\Big|_x - \sum_{\beta : j \in I_\beta} \partial_{u_\beta} f_i^\alpha(x,u) f_j^\beta(x,u) \equiv \sum_{k \in I_\alpha} c_{ij}^k(x) f_k^\alpha(x,u).$$

The Generalized Darboux theorem

A system:

$$\mathbf{r}_i(u_\alpha)\Big|_x = f_i^\alpha\Big(x, u(x)\Big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

with the data prescribed near a fixed point $\bar{x} \in \mathbb{R}^n$ by:

$$u_\alpha|_{\Xi_\alpha} = g_\alpha, \quad \alpha = 1, \ldots, m, \quad \text{where}$$

- $\mathbf{r}_1, \ldots, \mathbf{r}_n$ is a local C^1-frame near $\bar{x}$,
- $\Xi_\alpha \subset \mathbb{R}^n$ is an $(n - |I_\alpha|)$-dimensional manifold through $\bar{x}$, transversal to $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$,
- g_α is an arbitrary C^1-function on Ξ_α,
- $f_i^\alpha(x, u)$ are C^1-functions, satisfying near $(\bar{x}, g(\bar{x})) \in \mathbb{R}^n \times \mathbb{R}^m$,

  the integrability conditions on the previous page.

has a unique local C^1-solution near $\bar{x}$.

A particular case, when I_α's are the same for all α is treated in Benfield's thesis (2016).

Proof outline

1. Use Picard-type argument to prove existence and uniqueness of the solution $\tilde{u}$ of the restricted system, which

   - has the same equations and data as the original system

   - each equation is required to hold only for x on a certain, in general lower dimensional, submanifold of $\mathbb{R}^n$, containing $\bar{x}$.

   Integrability conditions are not used for this part!

2. Prove that $\tilde{u}$ is, in fact, a solution of the original system by:
   - introducing functions:

$$A_i^\alpha(x) = \mathbf{r}_i(\tilde{u}_\alpha)|_x - f_i^\alpha(x, \tilde{u}(x)), \qquad 1 \leq \alpha \leq m,\, i \in I_\alpha$$

   - using integrability conditions of the original system to show that functions $A_i^\alpha(x)$ satisfy a linear homogeneous system of equations of the "restricted type" covered by part 1.
   - observing that $A_i^\alpha(x) \equiv 0$ is a unique solution of such system.

Thank you!

Additional slides

More details on the "restricted system"

1. Let $W_i^t(x): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ denote the flow of $\mathbf{r}_i$:

$$\frac{d}{dt} W_i^t(x) = \mathbf{r}_i \Big|_{W_i^t(x)}.$$

2. For each α, choose an increasing order on the set of indices $I_\alpha = \{i_1, \ldots, i_{p(\alpha)}\}$ and define a map ρ from an appropriate open neighborhood of $(0, \bar{x})$ in $\mathbb{R}^p \times \Xi_\alpha$ to a neighborhood Ω of $\bar{x}$, by

$$\rho(t_1, t_2, \ldots, t_p, \xi) := W_{i_p}^{t_p} \cdots W_{i_2}^{t_2} W_{i_1}^{t_1} \xi.$$

3. By shrinking the domain of ρ, we can insure that $\Xi_\alpha^n = Im(\rho)$ is an open neighborhood of $\bar{x}$ with local C^1-coordinates $(\xi_1, \ldots, \xi_{n-p}, t_{i_1}, \ldots t_{i_p})$ and hence

$$\Xi_\alpha^k = \{x \in \Xi_\alpha^i \,|\, t_{i_{k+1}} = 0, \ldots, t_{i_p} = 0\}$$

are C^1-submanifolds of $\mathbb{R}^n$.

4. For the restricted system, we require that for each $\alpha = 1, \ldots, m$, and each $i_k \in I_\alpha$:

$$\mathbf{r}_{i_k}(u_\alpha)\Big|_x = f^\alpha_{i_k}(x, u(x)) \qquad \text{for } x \in \Xi^k_\alpha.$$

5. Picard-type argument implies that the fixed point $\tilde{u}$ of a contractive map:

$$\begin{aligned}\Phi[u]_\alpha(x) = g_\alpha(\xi) &+ \int_0^{t_1} f^\alpha_{i_1}\Big(W^b_{i_1}\xi,\, u(W^b_{i_1}\xi)\Big)\, db \\ &+ \int_0^{t_2} f^\alpha_{i_2}\Big(W^b_{i_2}W^{t_1}_{i_1}\xi,\, u(W^b_{i_2}W^{t_1}_{i_1}\xi)\Big)\, db \\ &\;\vdots \\ &+ \int_0^{t_p} f^\alpha_{i_p}\Big(W^b_{i_p}W^{t_{p-1}}_{i_{p-1}}\cdots W^{t_1}_{i_1}\xi,\, u(W^b_{i_p}W^{t_{p-1}}_{i_{p-1}}\cdots W^{t_1}_{i_1}\xi)\Big) db\end{aligned}$$

is the unique solution of the restricted system with the data

$$u_\alpha|_{\Xi_\alpha} = g_\alpha.$$

Applications

We encountered systems of the generalized Darboux type in our study of hyperbolic conservative systems:

$$U_t + F(U)_x = 0.$$

1. Jenssen, H. K., Kogan, I. A., Conservation laws with prescribed eigencurves. *J. of Hyperbolic Differential Equations (JHDE)* Vol. 7, No. 2., (2010) pp. 211– 254.

2. Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws *Communications in PDE's*, No. 37, (2012) , pp. 1096 – 1140.

At that time, we had to impose analyticity assumptions and use Cartan-Kähler theorem.

Conservation laws with prescribed eigencurves

Jacobian problem: Given a local frame $\mathcal{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_n\}$ on $\Omega \subset \mathbb{R}^n$, find all maps $f : \Omega \to \mathbb{R}^n$ such that $\mathcal{R}$ is the set of eigenvectors of the Jacobian matrix Df.

This leads to the λ-system, on the eigenvalues-to-be:

$$\begin{aligned} \mathbf{r}_i(\lambda^j) &= \Gamma^j_{ji}(\lambda^i - \lambda^j) && \text{for } i, j \in \{1, \dots, n\},\ i \neq j, \\ (\lambda^i - \lambda^k)\Gamma^k_{ji} &= (\lambda^j - \lambda^k)\Gamma^k_{ij} && \text{for } i, j, k \in \{1, \dots, n\}, \\ & && i < j,\ i \neq k,\ j \neq k. \end{aligned}$$

For $n > 2$ the size of the solution set depends on the properties of $\mathcal{R}$.

In a case when:
$\Gamma^1_{23} \neq 0$, but $c^1_{23} = 0$, $\Gamma^2_{31} = 0$, $\Gamma^3_{21} = 0$, $\Gamma^3_{31} = \Gamma^2_{21}$,

the λ-system reduces to $\lambda^2 = \lambda^3$ and a generalized Darboux-type system:

$$\begin{aligned} &\mathbf{r}_2(\lambda^1) = \Gamma^1_{12}(\lambda^2 - \lambda^1), && \mathbf{r}_1(\lambda^2) = \Gamma^2_{21}(\lambda^1 - \lambda^2) \\ &\mathbf{r}_3(\lambda^1) = \Gamma^1_{13}(\lambda^2 - \lambda^1), && \mathbf{r}_2(\lambda^2) = 0 \\ & && \mathbf{r}_3(\lambda^2) = 0 \end{aligned}$$

Extensions of systems of conservation laws

Hessian problem: Given a local frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ on $\Omega \subset \mathbb{R}^n$, find all functions $\eta \colon \Omega \to \mathbb{R}$, such that $\mathcal{R}$ is orthogonal with respect to the inner product defined by the Hessian matrix $D^2\eta$.

This leads to the β-system, on the "lengths" of vectors $\mathbf{r}_i$ with respect to $D^2\eta$:

$$
\begin{aligned}
r_i(\beta^j) &= \beta^j\,(\Gamma^j_{ij} + c^j_{ij}) - \beta^i\,\Gamma^i_{jj} && \text{for } i \neq j,\\
\beta^k\, c^k_{ij} + \beta^j \Gamma^j_{ik} - \beta^i\,\Gamma^i_{jk} &= 0 && \text{for } i < j,\ i \neq k,\ j \neq k,
\end{aligned}
$$

For $n > 2$ the size of the solution set depends on the properties of $\mathcal{R}$.

In a case when: $c^1_{23} \neq 0$, but $c^3_{12} = \Gamma^3_{12} = \Gamma^3_{21} \equiv 0$, $c^2_{13} = \Gamma^2_{13} = \Gamma^2_{31} \equiv 0$, and $\Gamma^2_{11} = \Gamma^3_{11} \equiv 0$,

the β-system reduces to $\beta^1 = 0$ and a generalized Darboux-type system:

$$
\begin{aligned}
\mathbf{r}_1(\beta^2) &= \beta^2\,(\Gamma^2_{12} + c^2_{12}), & \mathbf{r}_1(\beta^3) &= \beta^3\,(\Gamma^3_{13} + c^3_{13}),\\
\mathbf{r}_3(\beta^2) &= \beta^2\,(\Gamma^2_{32} + c^2_{32}) - \beta^3\,\Gamma^3_{22}, & \mathbf{r}_2(\beta^3) &= \beta^3\,(\Gamma^3_{23} + c^3_{23}) - \beta^2\,\Gamma^2_{33}.
\end{aligned}
$$