Systems of hyperbolic conservation laws with prescribed eigencurves

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Problem: Jacobians with prescribed eigenfields

Given: (i) A coordinate chart $\left(\Omega, u = (u^1, \ldots, u^n)\right)$ on $\mathbb{R}^n$;

(ii) n vector-fields $R_i(u) := (R_i^1(u), \ldots, R_i^n(u))^T$, $i = 1, \ldots, n$, independent over $\mathbb{R}$ at each point of Ω.

Find: a matrix-valued map $A \colon \mathcal{U} \to M_n$, where $\mathcal{U} \subset \Omega$ such that:

(i) $R_i(u)$, $i = 1, \ldots, n$ are right eigenvectors of $A(u)$ $\forall u \in \mathcal{U}$;

(ii) $A(u)$ is the Jacobian matrix of some map $f : \mathcal{U} \to \mathbb{R}^n$ relative to u-coordinates.

In other words:

Given: a local frame of vector fields $R_i(u) := (R_i^1(u), \ldots, R_i^n(u))^T$, $i = 1, \ldots, n$ on $\Omega \in \mathbb{R}^n$

Define: $R(u) := [R_1(u) \mid \cdots \mid R_n(u)]$,

$$L(u) := R(u)^{-1} = \begin{bmatrix} L^1(u) \\ \hline \vdots \\ \hline L^n(u) \end{bmatrix}.$$

Find: n smooth real-valued functions $\lambda^1(u), \ldots, \lambda^n(u)$ on a neighborhood $\mathcal{U} \subset \Omega$ s.t. with $\Lambda(u) := \mathrm{diag}[\lambda^1(u), \ldots, \lambda^n(u)]$

$$A(u) := R(u)\Lambda(u)L(u)$$

is the Jacobian matrix of some map $f : \mathcal{U} \to \mathbb{R}^n$ relative to u-coordinates.

How many solutions?

How many free constants and functions determine a general solution $\lambda^1(u), \ldots, \lambda^n(u)$?

Trivial solutions

- $\forall R_1(u), \ldots, R_n(u)$ $\exists$ one-parameter family of trivial solutions $\lambda^1(u) = \cdots = \lambda^n(u) \equiv \bar{\lambda}$, where $\bar{\lambda} \in \mathbb{R}$:

$$R(u)\,\bar{\Lambda}\,L(u) = \bar{\Lambda} = Df \text{ for } f = \bar{\lambda}u + \bar{u},\ \bar{u} \in \mathbb{R}^n.$$

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- $\exists R_1(u), \ldots, R_n(u)$ s.t. there are only trivial solutions. Example:

$$R_1 = [u^1,\ u^2,\ 0]^T,\ R_2 = [-u^2,\ u^1,\ 0]^T,\ R_3 = [-u^2,\ u^1,\ 1]^T.$$

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- $\lambda^1(u) = \cdots = \lambda^n(u)$ is a solution

$$\Updownarrow$$

$$\lambda^1(u) = \cdots = \lambda^n(u) \equiv \bar{\lambda} \text{ for some } \bar{\lambda} \in \mathbb{R}.$$

Scaling invariance

$\lambda^1(u), \ldots, \lambda^n(u)$ is a solution for vector-fields $R_1(u), \ldots, R_n(u)$

$$\Updownarrow$$

$\lambda^1(u), \ldots, \lambda^n(u)$ is a solution for $\tilde{R}_i = \alpha^i(u) R_i$, $i = 1, \ldots, n$ for any smooth functions $\alpha^i \colon \Omega \to \mathbb{R}$.

Proof:

$$R(u)\Lambda(u)L(u) \text{ is a Jacobian} \Leftrightarrow \tilde{R}(u)\Lambda(u)\tilde{L}(u) \text{ is a Jacobian.}$$

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---

we prescribe eigenfields-to-be up to a scaling

$$\Updownarrow$$

we prescribe eigencurves-to-be (integral curves of eigenfields)

System of conservation laws

$$u_t + f(u)_x = 0\,. \qquad (1)$$

- one space-dimension: $x \in \mathbb{R}$; one time-dimension: $t \in \mathbb{R}$.
- $u(x,t) \in \Omega \subset \mathbb{R}^n$ (n equations on n unknown state variables).
- nonlinear *flux* $f : \Omega \to \mathbb{R}^n$.

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$$LHS(1) = u_t + Df\, u_x$$

(1) is *hyperbolic* if $\forall u \in \Omega$ Jacobian $Df(u)$ is diagonalizable over $\mathbb{R}$.
(1) is *strictly hyperbolic* if $\forall u \in \Omega$ all eigenvalues of $Df(u)$ are real and distinct.

Riemann problem

$$u_t + f(u)_x = 0\,. \tag{1}$$

with a step function as an initial data at $t = 0$:

$$u_0(x) = \begin{cases} u_-\,, & x < 0 \\ u_+\,, & x > 0\,. \end{cases}$$

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Self-similar solutions $u(x,t) = \phi(\frac{x}{t})$ of Riemann problems, called *wave curves*, exist through each strictly hyperbolic state $\bar{u}$. They are locally made of two components with the second order contact at $\bar{u}$:

- rarefaction states that are part of eigencurves
- shock states that are part of Hugoniot locus $\{\, u \in \Omega \mid \exists\, s \in \mathbb{R} :\ f(u) - f(\bar{u}) = s \cdot (u - \bar{u}) \,\}$.

Direct Formulation

- A matrix $A(u) = (A^i_j(u))$ is a Jacobian on subset $\Omega \subset \mathbb{R}^n$ smoothly contractible to a point.

$$\frac{\partial A^i_j(u)}{\partial u^k} = \frac{\partial A^i_k(u)}{\partial u^j} \text{ for all } i, j, k = 1, \ldots, n \text{ with } j < k,$$

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- $A(u) = R(u)\Lambda(u)L(u)$ is a Jacobian

$$\Updownarrow$$

$$\sum_{m=1}^{n} \left[C^i_{mj}\partial_k\lambda^m - C^i_{mk}\partial_j\lambda^m + \lambda^m\left(\partial_k C^i_{mj} - \partial_j C^i_{mk}\right)\right] = 0\,,$$

$$i,j,k = 1,\dots,n \text{ with } j < k,$$

where

$$C^i_{mj}(u) := R^i_m(u)L^m_j(u) \qquad \text{(no summation)}, \qquad \partial_i = \frac{\partial}{\partial u_i}$$

$$\sum_{m=1}^{n}\Big[C^i_{mj}\partial_k\lambda^m - C^i_{mk}\partial_j\lambda^m + \lambda^m\big(\partial_k C^i_{mj} - \partial_j C^i_{mk}\big)\Big] = 0\,,$$

$$i,j,k = 1,\dots,n \;\text{ with }\; j<k$$

- A linear variable coefficient system of $\frac{n^2(n-1)}{2}$ of first order PDEs for n unknowns $\lambda^1,\dots,\lambda^n$.
- For $n \geq 3$ it is an overdetermined system.

Formulation in terms of differential forms

$$A(u) \text{ is a Jacobian matrix} \qquad \Longleftrightarrow \qquad dA(u) \wedge du = 0\,,$$

where $du := (du^1, \ldots, du^n)^T$.

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$A(u) = R(u)\Lambda(u)L(u)$ is a Jacobian

$$\Updownarrow$$

$$\{L(dR)\Lambda + d\Lambda - \Lambda L(dR)\} \wedge L du = 0\,.$$

(LHS is an n-vector of differential two-forms)

Rewriting in terms of the given frame:

- $r_i(u) := \sum_{m=1}^n R_i^m(u)\frac{\partial}{\partial u^m}$ is given frame
- $\ell^i(u) := \sum_{m=1}^n L_m^i(u)du^m$ is the dual coframe.
- $\ell := (\ell^1, \dots, \ell^n)^T$
- $\mu := R^{-1}dR = LdR$ matrix of one-forms

$$\big(L(dR)\Lambda + d\Lambda - \Lambda L(dR)\big) \wedge Ldu = 0$$

$$\Updownarrow$$

$$(\mu\Lambda + d\Lambda - \Lambda\mu) \wedge \ell = 0\,.$$

$$\Updownarrow$$

Algebraic-geometric system (the λ-system)

$$\left[\left(\mu\Lambda + d\Lambda - \Lambda\mu\right) \wedge \ell\right](r_i, r_j) = 0 \text{ for } 1 \leq i < j \leq n$$

$$\Updownarrow$$

$n(n-1)$ linear, homogeneous, 1st order PDEs and $\frac{n(n-1)(n-2)}{2}$ algebraic equations.

$$\begin{aligned} r_i(\lambda^j) &= \Gamma^j_{ji}(u)(\lambda^i - \lambda^j) && \text{for } i \neq j, \\ (\lambda^i - \lambda^k)\Gamma^k_{ji}(u) &= (\lambda^j - \lambda^k)\Gamma^k_{ij}(u) && \text{for } i < j,\ i \neq k,\ j \neq k, \end{aligned}$$

where $\Gamma^k_{ij} := L^k(DR_j)R_i$.

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where $\Gamma^k_{ij} := L^k(DR_j)R_i$.

$n = 2$ – no algebraic constraints. General solution depends on 2 arbitrary functions of 1 variable. (see Defermos)

Structure coefficients and connection components

- dual frame and coframe on Ω:

$$r_i := \sum_{m=1}^{n} R_i^m(u)\frac{\partial}{\partial u^m}\,, \quad \ell^i := \sum_{m=1}^{n} L^i_m(u)du^m.$$

$$[r_i, r_j] = \sum_{k=1}^{n} c_{ij}^k\, r_k, \quad d\ell^k = -\sum_{i<j} c_{ij}^k\,\ell^i \wedge \ell^j\,.$$

- $\Gamma_{ij}^k := L^k(DR_j)R_i$ is the Christoffel symbols of the connection $\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j} = 0$ computed relative to the frame $\{r_1, \ldots, r_n\}$ i.e

$$\nabla_{r_i} r_j = \sum_{k=1}^{n} \Gamma_{ij}^k r_k.$$

- Matrix $\mu := LdR$ of connection forms with $\mu_j^k = \sum_{i=1}^n \Gamma_{ij}^k \ell^i\,.$

Symmetry and flatness

$$d\ell = -\mu\wedge\ell \qquad \text{(Symmetry)}, \qquad d\mu = -\mu\wedge\mu \qquad \text{(Flatness)}.$$

$$\Updownarrow$$

$$c^i_{km} = \Gamma^i_{km} - \Gamma^i_{mk} \qquad \text{(Symmetry)}$$

and

$$r_m\left(\Gamma^j_{ki}\right) - r_k\left(\Gamma^j_{mi}\right) = \sum_{s=1}^{n}\left(\Gamma^j_{ks}\Gamma^s_{mi} - \Gamma^j_{ms}\Gamma^s_{ki} - c^s_{km}\Gamma^j_{si}\right) \ \text{(Flatness)}.$$

Rich frame

- **Definition** A frame $r_1, \ldots, r_n$ is *rich* if each pair of vector-fields is in involution, i. e. $\forall 1 \leq i, j \leq n$:

$$[r_i, r_j] = c_{ij}^i r_i + c_{ij}^j r_j \quad \Leftrightarrow \quad c_{ij}^k = 0 \quad k \neq i,\ k \neq j.$$

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$$\Downarrow$$

- $\exists$ smooth functions $\alpha^i : \Omega \to \mathbb{R}$, $i = 1, \dots, n$ such that $\tilde{r}_1 := \alpha^1(u) r_1, \dots, \tilde{r}_n := \alpha^n r_n$ commute.

$$\Downarrow$$

Riemann invariants

- $\exists$ a change of coordinates

$$(w^1(u), \ldots, w^n(u)) = \rho(u)$$

s.t. $\tilde{r}_i = \frac{\partial}{\partial w^i}, \quad i = 1, \ldots, n.$

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- the dual coframe: $\tilde{\ell}^i = dw^i, \quad i = 1, \dots, n.$
- Coordinates $w^1(u), \dots, w^n(u)$ are called *Riemann invariants.*

λ-system in Riemann Invariants $(w^1(u), \ldots, w^n(u)) = \rho(u)$

$$\partial_i \kappa^j = Z^j_{ji}(\kappa^i - \kappa^j) \quad \text{for} \quad 1 \leq i \neq j \leq n,$$
$$Z^k_{ij}(\kappa^j - \kappa^i) = 0 \quad \text{for} \quad 1 \leq k \neq i < j \neq k \leq n,$$

where $\partial_i = \frac{\partial}{\partial w^i}$ and

$$\kappa^i(w) := \lambda^i \circ \rho^{-1}(w) \quad \text{and} \quad Z^k_{ij}(w) := \Gamma^k_{ij} \circ \rho^{-1}(w).$$

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$$Z^i_{km} = Z^i_{mk} \quad \text{(Symmetry)}$$

$$\partial_m\left(Z^j_{ki}\right) - \partial_k\left(Z^j_{mi}\right) = \sum_{s=1}^{n}\left(Z^j_{ks} Z^s_{mi} - Z^j_{ms} Z^s_{ki}\right) \qquad \text{(Flatness).}$$

Sévennec's problem:

For a given quasilinear system

$$v_t + A(v)v_x = 0\,,$$

Sévennec shows that there is a coordinate system in which the system is conservative if and only if there exists a flat and symmetric affine connection ∇ such that its Christoffel symbols and the eigenvalues of $A(u)$ satisfy

$$\begin{aligned} r_i(\lambda^j) &= \Gamma^j_{ji}(\lambda^i - \lambda^j) && \text{for } i \neq j, \\ (\lambda^i - \lambda^k)\Gamma^k_{ji} &= (\lambda^j - \lambda^k)\Gamma^k_{ij} && \text{for } i < j,\ i \neq k,\ j \neq k. \end{aligned}$$

Solution strategy

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- Express some λ's in terms of the others from algebraic equations.
- Substitute in differential equations.
- Use integrability theorems (Frobenius, Darboux, Cartan-Kähler) to describe the set of solutions.

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Flatness and symmetry of the connection play essential role in checking compatibility conditions.

Algebraic constraints

- $\frac{n(n-1)(n-2)}{2}$ linear equations:

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- $n-1$ variables: $x^k := \lambda^k - \lambda^1\,, \qquad k = 2, \dots, n.$

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- $n-1$ variables: $x^k := \lambda^k - \lambda^1\,, \qquad k = 2, \ldots, n.$
- matrix formulation:

$$Nx = 0, \text{ where}$$

  - x is the $(n-1)$-vector $(x^2, \ldots, x^n)^T$
  - N is $\frac{n(n-1)(n-2)}{2} \times (n-1)$-matrix with entries that are either zero, $\pm\Gamma^k_{ij}$, or $\Gamma^k_{ij} - \Gamma^k_{ji} = c^k_{ij}$ for some $i \neq j \neq k \neq i$.

Extreme cases

- $$\operatorname{rank} N = n-1 \quad \Rightarrow \quad x^k = 0\,, \quad k = 2, \dots, n$$

$$\Downarrow$$

$$\text{only trivial solutions } \lambda^1 = \cdots = \lambda^n \equiv \bar{\lambda} \in \mathbb{R}.$$

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- $$\text{rank}\, N = 0 \quad \Leftrightarrow \quad \Gamma^k_{ij} = 0\,, \forall i \neq j \neq k \neq i$$

$$\Downarrow$$

$$c^k_{ij} = 0\,, \forall i \neq j \neq k \neq i \quad \Leftrightarrow \quad \{r_1, \dots, r_n\} \text{ is rich.}$$

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Remarks

- only trivial solutions $\not\Rightarrow$ $\operatorname{rank}(N) = n-1$.
- $\{r_1, \ldots, r_n\}$ is rich $\not\Rightarrow$ $\operatorname{rank}(N) = 0$.
- we will show:
  $\{r_1, \ldots, r_n\}$ is rich and admits strictly hyperbolic solutions
  $\Rightarrow$ $\operatorname{rank}(N) = 0$.

Algebraic equations for $n = 3$

$$Nx = \begin{bmatrix} \Gamma^1_{32} & -\Gamma^1_{23} \\ (\Gamma^2_{31} - \Gamma^2_{13}) & \Gamma^2_{13} \\ \Gamma^3_{12} & (\Gamma^3_{21} - \Gamma^3_{12}) \end{bmatrix} \begin{bmatrix} x^2 \\ x^3 \end{bmatrix} = 0\,,$$

where $x^2 = \lambda^2 - \lambda^1$ and $x^3 = \lambda^3 - \lambda^1$.

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- rank $N = 2 \quad \Rightarrow \quad \lambda$-system has only trivial solutions $\lambda^1 = \lambda^2 = \lambda^3 = \bar{\lambda} \in \mathbb{R}$
- rank $N = 0 \quad \Rightarrow \quad \{r_1, r_2, r_3\}$ is rich. Darboux theorem $\Rightarrow$ general solution depends on 3 arbitrary functions of one variable.

$n = 3 \quad \text{rank}\, N = 1$

There is a unique (up to non-vanishing scalings) relation:

$$\alpha_1\lambda^1 + \alpha_2\lambda^2 + \alpha_3\lambda^3 = 0\,, \text{ where } \alpha_3 = -(\alpha_1 + \alpha_2)$$

Sub-cases: the algebraic relation involves

(i) all three λ^i with non-zero coefficients

(ii) only two of three λ^i with non-zero coefficients (after possible permutation of indices $\alpha_1 = 0$) $\quad \Rightarrow \quad \lambda^2 = \lambda^3$ (no strictly hyperbolic solutions to λ-system in this case.)

λ-system for $n = 3$ rank $N = 1$ sub-case (i)

- (after possible permutation of indices) $c^1_{32} \neq 0$, $\Gamma^1_{32} \neq 0$, $\Gamma^1_{23} \neq 0$ and

$$\lambda^1 = \frac{1}{c^1_{32}}(\Gamma^1_{32}\lambda^2 - \Gamma^1_{23}\lambda^3),$$

- Substitution in 6 PDE's of the λ-system produces Frobenius system

$$r_i(\lambda^s) = \phi^s_i(u)(\lambda^2 - \lambda^3) \qquad \text{for } s = 2, 3 \text{ and } i = 1, 2, 3,$$

where ϕ^s_i are known functions of $\Gamma(u)$'s.

λ-system for $n = 3$ rank $N = 1$ sub-case (i)

Compatibility conditions arise by substituting directional derivatives of $r_i(\lambda^s)$'s given by PDE's into

$$[r_i, r_j]\lambda^s = \sum_{k=1}^{3} c_{ij}^k r_k \lambda^s, \quad s = 2, 3; \quad 1 \leq i < j \leq 3.$$

- If they hold identically then the general solution depends on two arbitrary constants $\bar{\lambda}^2, \bar{\lambda}^3 \in \mathbb{R}$, s.t. for $\bar{u} \in \Omega$:

$$\lambda^2(\bar{u}) = \bar{\lambda}^2 \text{ and } \lambda^3(\bar{u}) = \bar{\lambda}^3.$$

λ-system for $n = 3$ rank $N = 1$ sub-case (i)

Compatibility conditions arise by substituting directional derivatives of $r_i(\lambda^s)$'s given by PDE's into

$$[r_i, r_j]\lambda^s = \sum_{k=1}^{3} c_{ij}^k r_k \lambda^s, \quad s = 2, 3; \quad 1 \le i < j \le 3.$$

- If they hold identically then the general solution depends on two arbitrary constants $\bar{\lambda}^2, \bar{\lambda}^3 \in \mathbb{R}$, s.t. for $\bar{u} \in \Omega$:

$$\lambda^2(\bar{u}) = \bar{\lambda}^2 \text{ and } \lambda^3(\bar{u}) = \bar{\lambda}^3.$$

- Otherwise, there are only trivial solutions:

$$\lambda^1(u) = \lambda^2(u) = \lambda^3(u) \equiv \bar{\lambda} \in \mathbb{R}.$$

λ-system for $n = 3$ rank $N = 1$ sub-case (ii)

- algebraic equations $\Rightarrow$ $\lambda^2(u) = \lambda^3(u) =: h(u)$ (up to permutation of indices).
- substitution into 6 PDE's $r_i(\lambda^j) = \Gamma^j_{ji}(\lambda^i - \lambda^j)\,,\quad 1 \leq i \neq j \leq 3$ produces:

$$\begin{aligned} r_1(h) &= \Gamma^2_{21}(\lambda^1 - h), & r_2(h) &= 0, \\ r_1(h) &= \Gamma^3_{31}(\lambda^1 - h), & r_3(h) &= 0, \end{aligned}$$

$$\begin{aligned} r_2(\lambda^1) &= \Gamma^1_{12}(h - \lambda^1), \\ r_3(\lambda^1) &= \Gamma^1_{13}(h - \lambda^1). \end{aligned}$$

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- If $\Gamma^2_{21} \neq \Gamma^3_{31}$, then
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- If $\Gamma^2_{21} = \Gamma^3_{31}$ Cartan-Kähler Thm $\Rightarrow$ general solution depends on one constant (value of h at $\bar u \in \Omega$) and one function of one variable (values of λ^1 along a curve)

Rich frame

$\exists$ a change of coordinates $u \mapsto \rho(u) = (w^1(u), \ldots, w^n(u))$
s.t. λ-system becomes

$$\partial_i \kappa^j = Z^j_{ji}(\kappa^i - \kappa^j) \quad \text{for} \quad 1 \le i \ne j \le n,$$
$$Z^k_{ij}(\kappa^j - \kappa^i) = 0 \quad \text{for} \quad 1 \le k \ne i < j \ne k \le n,$$

where $\partial_i = \frac{\partial}{\partial w^i}$ and
$\kappa^i(w) := \lambda^i \circ \rho^{-1}(w)$ and $Z^k_{ij}(w) := \Gamma^k_{ij} \circ \rho^{-1}(w)$.

- $\forall$ distinct i, j, k: $Z^k_{ij} = 0 \quad \Rightarrow \quad$ no algebraic constraints
- $\exists$ distinct i, j, k s.t. $Z^k_{ij} \ne 0 \quad \Rightarrow \quad$ multiplicity conditions on eigenvalues are implied by the system.

Rich frame. No algebraic constraints.

$$\partial_i \kappa^j = Z^j_{ji}(\kappa^i - \kappa^j) \text{ for } 1 \leq i \neq j \leq n, \quad \partial_i := \frac{\partial}{\partial w_i}.$$

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- Compatibility conditions $\partial_k \partial_m \kappa^j = \partial_m \partial_k \kappa^j$, where the first derivatives $\partial_i \kappa^j$, $i = 1, \ldots, n$ are given by the equations, are met due to the flatness of the connection.

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- Darboux theorem $\Rightarrow$ general solution depends on n functions of one variable $\phi^i(w^i)$, $i = 1, \ldots, n$ s.t. for $\bar{w} \in \Omega$

$$\kappa^i(\bar{w}^1, \ldots, \bar{w}^{i-1}, w^i, \bar{w}^{i+1}, \ldots, \bar{w}^n) = \phi^i(w^i).$$

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- all $n = 2$ frames belong to this case.
- rich orthogonal frames belong to this case.

Rich system with non-trivial algebraic constraints

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$$Z^k_{ij}(\kappa^j - \kappa^i) = 0 \qquad \text{for} \quad 1 \le k \ne i < j \ne k \le n.$$

- $\exists$ distinct i, j, k s.t. $Z^k_{ij} \ne 0$

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- multiplicity conditions on eigenvalues are implied by the algebro-differential system (no strictly hyperbolic conservation laws in this case).
- Darboux theorem $\Rightarrow$ general solution depends on s_0 constants and s_1 functions of one variable, where
  - s_0 is the number of distinct eigenvalues of multiplicity > 1,
  - s_1 is the number of eigenvalues of multiplicity 1.

The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

$v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, $p(v, S) > 0$ is pressure as a given function of v and S, s.t $p_v < 0$.

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- $U_t + f(U)_x = 0$,where $U = (v, u, S)$ and $f(U) = (-u, p(v, S), 0)$.

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- $U_t + f(U)_x = 0$,where $U = (v,u,S)$ and $f(U) = (-u, p(v,S), 0)$.
- eigenvalues of Df are $\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}.$
- eigenvectors of D_F are $R_1 = [1,\ \sqrt{-p_v},\ 0]^T$, $R_2 = [-p_S,\ 0,\ p_v]^T$, $R_3 = [1,\ -\sqrt{-p_v},\ 0]^T$

Inverse Problem

- For a given pressure function $p = p(v, S) > 0$, with $p_v < 0$, and vector fields $R_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $R_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $R_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$ determine the class of conservative systems with these as eigenfields by solving the λ-system for λ^1, λ^2, λ^3.

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- Observation: frame is rich $\Leftrightarrow$ $\left(\frac{p_S}{p_v}\right)_v \equiv 0$

λ-system:

- differential equations

$$\begin{aligned}
r_1(\lambda^2) &= 0 \\
r_1(\lambda^3) &= \frac{p_{vv}}{4p_v}(\lambda^3 - \lambda^1) \\
r_2(\lambda^1) &= \frac{p_v}{2}\left(\frac{p_S}{p_v}\right)_v(\lambda^1 - \lambda^2) \\
r_2(\lambda^3) &= \frac{p_v}{2}\left(\frac{p_S}{p_v}\right)_v(\lambda^3 - \lambda^2) \\
r_3(\lambda^1) &= \frac{p_{vv}}{4p_v}(\lambda^1 - \lambda^3) \\
r_3(\lambda^2) &= 0\,.
\end{aligned}$$

- one independent algebraic equation:

$$\frac{p_v}{4}\left(\frac{p_S}{p_v}\right)_v(\lambda^1 + \lambda^3 - 2\lambda^2) = 0.$$

- Rich frame $\Leftrightarrow$ $\left(\frac{p_S}{p_v}\right)_v \equiv 0$ $\Leftrightarrow$ no algebraic constraints.

Solution in the non-rich case:

- this is $n = 3$ case with one algebraic constraint $\lambda^1 + \lambda^3 = 2\lambda^2$ that involves all three λ's $\Rightarrow$ the general solution depends on two constants.

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- from the differential part of λ-system we obtain:

$$\lambda^1 = \bar{\lambda} - C\sqrt{-p_v}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = \bar{\lambda} + C\sqrt{-p_v}\,.$$

Solution in the rich case $\left(\frac{p_S}{p_v}\right)_v \equiv 0$

- this is rich case with no algebraic constraints $\Rightarrow$ solution depends on 3 arbitrary functions in one variable.
- $\left(\frac{p_S}{p_v}\right)_v \equiv 0 \quad \Leftrightarrow \quad p(v,S) = \Pi(\xi)$, where $\xi = v + F(S)$.
- from the differential part of λ-system we obtain:

$$\lambda^2 = \lambda^2(S), \quad \lambda^1 = A(\xi,u), \quad \lambda^3 = B(\xi,u),$$

where

$$A_\xi - \sqrt{-\Pi'(\xi)}A_u = a\,(B-A)\,,\; B_\xi + \sqrt{-\Pi'(\xi)}B_u = a\,(A-B)$$

and $a = -\frac{p_{vv}}{4p_v}$.