Object-image correspondence under projections

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Projection problem:

Given: A subset $\mathcal{Z}$ in $\mathbb{R}^3$ and a subset $\mathcal{X}$ in $\mathbb{R}^2$.

Decide: whether there exists a projection $P\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $\mathcal{X} = P(\mathcal{Z})$

Main focus: $\mathcal{Z}$ and $\mathcal{X}$ are rational algebraic curves.

Digression:

- $\mathcal{Z}$ and $\mathcal{X}$ are non-rational algebraic curves.
- $\mathcal{Z}$ and $\mathcal{X}$ are finite lists of points.

Motivation: Establishing a correspondence between objects in 3D and their images, when camera parameters and position are unknown.

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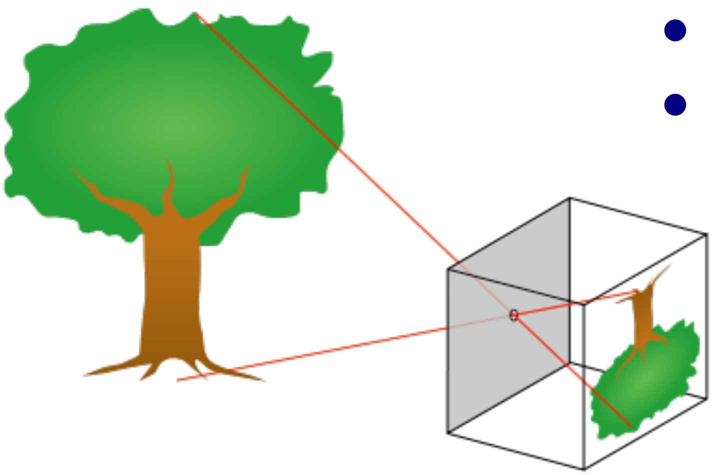
11 **degrees of freedom:**



- location of the center (3 parameters);
- position of the image plane (3 parameters);
- choice of, in general, non-orthogonal, coordinates on the image plane (5 parameters, since the overall scale is absorbed by the choice of the distance between the image plane and the center).

$P : \mathbb{R}^3 \to \mathbb{R}^2$

$$x = \frac{p_{11}\, z_1 + p_{12}\, z_2 + p_{13}\, z_3 + p_{14}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}},$$
$$y = \frac{p_{21}\, z_1 + p_{22}\, z_2 + p_{23}\, z_3 + p_{24}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}}.$$

Approaches

Direct: Given $\mathcal{Z} \subset \mathbb{R}^3$ and $\mathcal{X} \subset \mathbb{R}^2$, set up a system of conditions on the projection parameters and then check whether or not this system has a solution.

- Hartley and Zisserman (2004) for finite lists of points.
- Feldmar, Ayache, and Betting (1995) for images of curves and surfaces taken by cameras with known internal parameters (central projections with 6 unknown parameters) under additional assumption on the image curves.

Implicit: Establish necessary and sufficient conditions that $\mathcal{Z}$ and $\mathcal{X}$ must satisfy in order for a projection to exist.

- Arnold, Stiller, and Sturtz (2006, 2007) for finite lists of points defined an algebraic variety that characterizes pairs of sets related by a parallel projection.

Our approach is in between of “direct” and “implicit” approach. We exploit the relationship between the projection problem and equivalence problem under group-actions to find the conditions that has to be satisfied by the object, the image and the center of the projection:

1. We reduce the projection problem for curves to a modification of a group equivalence problem (group-equivalence of a given planar curves to a curve from 3-parameter family of planar curves).

2. The group-equivalence problem is solved by computing signatures, based on a separating set of rational differential invariants.

Comparison of the "direct" and "signature" approach for central projection

Direct approach: Given two rational maps $\Gamma\colon \mathbb{R} \to \mathbb{R}^3$ and $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, parametrizing algebraic curves $\mathcal{Z} \subset \mathbb{R}^3$ and $\mathcal{X} \subset \mathbb{R}^2$, determine the truth of the statement:

$$\exists P \in \mathbb{R}^{3\times 4} \quad \det(p_{ij})_{i=1\ldots 3}^{j=1\ldots 3} \neq 0$$
$$\forall s \text{ in the domain of } \Gamma(s) \quad \exists t \in \mathbb{R} \quad \gamma(t) = P(\Gamma(s)).$$

Signature approach: Given two rational maps $S_{\mathcal{X}}\colon \mathbb{R} \to \mathbb{R}^2$ and $S_{\mathcal{Z}}\colon \mathbb{R}^4 \to \mathbb{R}^2$, determine the truth of the statement:

$$\exists c \in \mathbb{R}^3$$
$$\forall s \text{ in the domain of } S_{\mathcal{Z}}(c,s) \quad \exists t \in \mathbb{R} \quad S_{\mathcal{Z}}(c,s) = S_{\mathcal{X}}(t).$$

- Real quantifier elimination problems are known to have algorithmic solutions of high complexity.
- In general, the less parameters to eliminate – the better (although other factors may be important).

Remarks:

- Our approach can be used for finite lists of points (with signatures based on a separating set of algebraic invariants)
- The projcetion problem can be considered over $\mathbb{C}$ and the proposed method is easier to implement over $\mathbb{C}$.
- The same method applies for projections of curves from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ (but an implementation are much harder) and, in principle, for projection of higher dimensional objects.

Projective camera model

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix}$$

- [] denotes an equivalence class of a matrix/ a vector under scalings by non-zero constants.
- $\mathbb{R}^n \hookrightarrow \mathbb{P}^n$: $\mathbf{z} = (z_1, \ldots, z_n) \to [z_1, \ldots, z_n, 1] = [\mathbf{z}]$.
- points $[z_1, \ldots, z_n, 0] \in \mathbb{P}^n$ are said to be at infinity
- P is 3×4 matrix of rank 3
- $[P]: \mathbb{P}^3 \to \mathbb{P}^2$ is undefined at a point $[z_1^0, z_2^0, z_3^0, z_4^0] \in \mathbb{P}^3$ such that $P\,(z_1^0, z_2^0, z_3^0, z_4^0)^T = (0, 0, 0)^T$.
- $[z_1^0, z_2^0, z_3^0, z_4^0] \in \mathbb{P}^3$ is the camera center (the center of projection).

Types of cameras:

finite if its center is not at $\infty \Leftrightarrow$ left 3×3 submatrix of P is non-singular (central projections with 11 degrees of freedom);

infinite center is at ∞;

affine center is at ∞ and the preimage of the line at ∞ in $\mathbb{P}^2$ is the plane at infinity in $\mathbb{P}^3 \Leftrightarrow$ the last row of $[P]$ is $[0, 0, 0, 1]$, (parallel projections with 8 degrees of freedom).

Standard cameras:

finite: projection centered at the origin to the plane $z_3 = 1$:

$$[P^0_{\mathcal{C}}] := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix};$$

affine: orthogonal projection to $z_1 z_2$-plane: $[P^0_{\mathcal{P}}] := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Projection of algebraic curves

Fact: For an algebraic curve $\mathcal{Z} \subset \mathbb{R}^n$ there exists unique projective closure $[\mathcal{Z}] \subset \mathbb{P}^n$.

Definition: We say that $\mathcal{Z} \subset \mathbb{R}^3$ projects to $\mathcal{X} \subset \mathbb{R}^2$ if $\exists [P]$ s.t. $\overline{[P][\mathcal{Z}]} = [\mathcal{X}]$,*

where $[P][\mathcal{Z}] := \{[P][\mathbf{z}] \mid [\mathbf{z}] \in [\mathcal{Z}]\}$. †

Notation: $\mathcal{X} = P(\mathcal{Z})$

Example: $\mathcal{Z} \in \mathbb{R}^3$ defined by equations $\left\langle z_1 = z_3^2,\, z_2 = z_3^4 \right\rangle$ projects to the curve $\mathcal{X} \in \mathbb{R}^2$ defined by equation $\left\langle y = x^2 \right\rangle$ by orthogonal projection $P^0_{\mathcal{P}}$, but $P^0_{\mathcal{P}}[\mathcal{Z}]$ occupies only "half" of the image.

* bar denotes algebraic closure.

† If $P[\mathbf{z}] = 0$, then the image of $[\mathbf{z}]$ is undefined.

Groups

projective: $\mathcal{PGL}(n+1) = \{[B] \mid B \in \mathcal{GL}(n+1)\}$

affine: $\mathcal{A}(n) = \{[B] \mid B \in \mathcal{GL}(n+1)$, the last row of B is $(0,\ldots,0,1)\}$.

equi-affine: $\mathcal{SA}(n) = \{[B] \mid B \in \mathcal{SL}(n+1)$, the last row of B is $(0,\ldots,0,1)\}$.

... and their actions:

$\mathcal{PGL}(n+1)$ and its subgroups act

- on $\mathbb{P}^n$ by $([B],[z_1,\ldots,z_n,z_0]^T) \to [B]\,[z_1,\ldots,z_n,z_0]^T$.

$$\Downarrow$$

- on $\mathbb{R}^n$ by linear fraction transformations.

Actions on cameras

finite cameras = the set of central projections

$$\mathcal{CP} = \Big\{ [P] \, | P \text{ is a } 3 \times 4 \text{ matrix whose left } 3 \times 3 \text{ minor is non-zero} \Big\}.$$

affine cameras = the set of parallel projections

$$\mathcal{PP} = \Big\{ [P] \, | P \text{ is a } 3 \times 4 \text{ matrix of rank } 3 \text{ whose last row is } (0,0,0,1) \Big\}.$$

Proposition: $\boxed{[P] \to [A]\,[P]\,[B^{-1}]}$ defines a transitive action

- of $\mathcal{PGL}(3) \times \mathcal{A}(3)$ on $\mathcal{CP}$.
- of $\mathcal{A}(2) \times \mathcal{A}(3)$ on $\mathcal{PP}$.

Proof of the transitivity of $\mathcal{PGL}(3) \times \mathcal{A}(3)$-action on $\mathcal{CP}$

- If $[P] \in \mathcal{CP}$ then P is 3×4 matrix whose left 3×3 submatrix is non-singular $\Rightarrow \exists\, c_1, c_2, c_3 \in \mathbb{R}$ s. t.
  $p_{*4} = c_1\, p_{*1} + c_2\, p_{*2} + c_3\, p_{*3}$, where p_{*j} is the j-th column of P.*

- Define A to be the left 3×3 submatrix of P and $B := \begin{pmatrix} 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

- Observe that $[A] \in \mathcal{PGL}(3)$ and $[B] \in \mathcal{A}(3)$

- $\boxed{P = A\, P^0_{\mathcal{C}}\, B^{-1}}$, where $P^0_{\mathcal{C}} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

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*$(-c_1, -c_2, -c_3)$ is the center of P.

Proof of the transitivity of $\mathcal{A}(2) \times \mathcal{A}(3)$ action on $\mathcal{PP}$

- $\forall [P] \in \mathcal{PP}$ can be represented by a rank 3 matrix:
$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Exist $1 \leq i < j \leq 3$ such that the rank of the submatrix $\begin{pmatrix} p_{1i} & p_{1j} \\ p_{2i} & p_{2j} \end{pmatrix}$ is 2. For $1 \leq k \leq 3$, such that $k \neq i$ and $k \neq j$, there exist $c_1, c_2 \in \mathbb{R}$, such that $\begin{pmatrix} p_{1k} \\ p_{2k} \end{pmatrix} = c_1 \begin{pmatrix} p_{1i} \\ p_{2i} \end{pmatrix} + c_2 \begin{pmatrix} p_{1j} \\ p_{2j} \end{pmatrix}$.

- Define $A := \begin{pmatrix} p_{1i} & p_{1j} & p_{14} \\ p_{2i} & p_{2j} & p_{24} \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{A}(2)$ and $B \in \mathcal{A}(3)$ to have columns
$$b_{*i} := (1,0,0,0)^T, b_{*j} := (0,1,0,0)^T, b_{*k} := (-c_1, -c_2, 1, 0)^T, b_{*4} = (0,0,0,1)^T.$$

- $\boxed{P = A\, P^0_{\mathcal{P}}\, B^{-1}}$, where $P^0_{\mathcal{P}} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Group equivalence of curves:

Definition: Let $G \subset \mathcal{PGL}(n+1)$. We say that $\mathcal{X}_1 \subset \mathbb{R}^n$ is G-equivalent to $\mathcal{X}_2 \subset \mathbb{R}^n$ if $\exists [A] \in G$ s.t.

$$[\mathcal{X}_2] = \{[A][\mathbf{x}] \mid [\mathbf{x}] \in [\mathcal{X}_1]\}$$

Notation: $\mathcal{X}_1 = A(\mathcal{X}_2)$ or $\mathcal{X}_1 \cong_G \mathcal{X}_2$.

Camera decomposition implies:

(i) If $\mathcal{Z} \subset \mathbb{R}^3$ projects to $\mathcal{X} \subset \mathbb{R}^2$ by a parallel projection, then any curve that is $\mathcal{A}(3)$-equivalent to $\mathcal{Z}$ projects to any curve that is $\mathcal{A}(2)$-equivalent to $\mathcal{X}$ by a parallel projection.*

(ii) If $\mathcal{Z} \subset \mathbb{R}^3$ projects to $\mathcal{X} \subset \mathbb{R}^2$ by a central projection then any curve in $\mathbb{R}^3$ that is $\mathcal{A}(3)$-equivalent to $\mathcal{Z}$ projects to any curve on $\mathbb{R}^2$ that is $\mathcal{PGL}(3)$-equivalent to $\mathcal{X}$ by a central projection.†

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*It is not true that any two images of $\mathcal{Z}$ by $P_1, P_2 \in \mathcal{PP}$ are $\mathcal{A}(2)$-equivalent.

†It is not true that any two images of $\mathcal{Z}$ by $P_1, P_2 \in \mathcal{CP}$ are $\mathcal{PGL}(3)$-equivalent, but if P_1 and P_2 have the same center images are $\mathcal{PGL}(3)$-equivalent

Projection criteria for algebraic curves

($\mathcal{CP}$) A curve $\mathcal{Z} \subset \mathbb{R}^3$ projects onto a curve $\mathcal{X} \subset \mathbb{R}^2$ by a finite projection if and only if $\exists\, c_1, c_2, c_3 \in \mathbb{R}$ such that $\mathcal{X}$ is $\mathcal{PGL}(3)$-equivalent to a planar curve

$$\tilde{\mathcal{Z}}_{c_1,c_2,c_3} = \overline{\left\{ \left(\frac{z_1 + c_1}{z_3 + c_3}, \quad \frac{z_2 + c_2}{z_3 + c_3} \right) \,\middle|\, (z_1, z_2, z_3) \in \mathcal{Z} \right\}} \tag{1}$$

Remark: the projection center is $(-c_1, -c_2, -c_3)$.

($\mathcal{PP}$) An curve $\mathcal{Z} \subset \mathbb{R}^3$ projects onto an curve $\mathcal{X} \subset \mathbb{R}^2$ by an affine projection if and only if there $\exists\, c_1, c_2 \in \mathbb{R}$ and an ordered triplet $(i, j, k) \in \{(1,2,3),\ (1,3,2),\ (2,3,1)\}$ such that $\mathcal{X}$ is $\mathcal{A}(2)$-equivalent to

$$\tilde{\mathcal{Z}}^{i,j,k}_{c_1,c_2} = \overline{\left\{ \left(z_i + c_1\, z_k, \quad z_j + c_2\, z_k \right) \,\middle|\, (z_1, z_2, z_3) \in \mathcal{Z} \right\}} \tag{2}$$

Remark: this criterion can be reduced by considering non-overlaping cases.

Group-equivalence for planar curves.

Problem: Given a rational action of G on $\mathbb{R}^2$ and curves $\mathcal{X}_1 \subset \mathbb{R}^2$ and $\mathcal{X}_2 \subset \mathbb{R}^2$ decide whether there exists $g \in G$ such that $\mathcal{X}_1 = g(\mathcal{X}_2)$.

Solution: is an algebraic adaptation of a known local solution from differential geometry

- Find a separating set of two rational differential invariants.
- Use these invariants to define signatures $\mathcal{S}_{\mathcal{X}_1} \subset \mathbb{R}^2$ and $\mathcal{S}_{\mathcal{X}_2} \subset \mathbb{R}^2$.
- Prove that $\boxed{\mathcal{X}_1 \cong_G \mathcal{X}_2 \quad \Longleftrightarrow \quad \mathcal{S}_{\mathcal{X}_1} = \mathcal{S}_{\mathcal{X}_2}.}$

Prolongation of an action

A rational action G on $\mathbb{R}^2$ prolongs to an action the jet space $J^n = \mathbb{R}^{n+2}$ with coordinates $(x, y, y^{(1)}, \ldots, y^{(n)})$ as follows:

For a fixed $g \in G$ let $(\bar{x}, \bar{y}) = g \cdot (x, y)$, then $\bar{x}, \bar{y}$ are rational functions of (x, y). Then

$$g \cdot (x, y, y^{(1)}, \ldots, y^{(n)}) := (\bar{x}, \bar{y}, \bar{y}^{(1)}, \ldots, \bar{y}^{(n)}), \text{ where}$$

$$\bar{y}^{(1)} = \frac{\frac{d}{dx}\left[\bar{y}(x, y)\right]}{\frac{d}{dx}\left[\bar{x}(x, y)\right]} \text{ and for } k = 1, \ldots, n-1$$

$$\bar{y}^{(k+1)} = \frac{\frac{d}{dx}\left[\bar{y}^{(k)}(x, y, y^{(1)}, \ldots, y^{(k)})\right]}{\frac{d}{dx}\left[\bar{x}(x, y)\right]}.$$

$\frac{d}{dx}$ is the total derivative, applied under assumption that y is function of x. We note the duality of our view of variables $y^{(k)}$. On one hand, they are viewed as independent coordinate functions on J^n. On the other hand, operator $\frac{d}{dx}$ is applied under assumption that y is a function of x and, therefore, $y^{(k)}$ is also viewed as the k-th derivative of y with respect to x.

Invariants

- Let G act rationally on $\mathbb{R}^N$. A rational function $\Phi \colon \mathbb{R}^N \to \mathbb{R}$ is invariant if
$$\Phi(g \cdot w) = \Phi(w)$$
for all $w \in \mathbb{R}^n$ and $g \in G$, such that w and $g \cdot w$ are in the domain of Φ.

- A set $\mathcal{I}$ of rational invariants is separating on a subset $W \subset \mathbb{R}^N$ if W is contained in the domain of definition of each $\Phi \in \mathcal{I}$ and $\forall w_1, w_2 \in W$
$$\Phi(w_1) = \Phi(w_2),\ \forall \Phi \in \mathcal{I} \iff \exists g \in G \text{ such that } w_1 = g \cdot w_2.$$

- A function Φ on J^n is called a differential function. *
A differential function which is invariant under prolonged action of G on $\mathbb{R}^2$ is called a differential invariant.

- Let $\dim G = r$. Let K and T be rational differential invariants of orders $r-1$ and r, respectively. The set $\mathcal{I} = \{K, T\}$ is called differentially separating if $\{K\}$ is separating on a Zariski open subset $W^{r-1} \subset J^{r-1}$ and $\mathcal{I} = \{K, T\}$ is separating on a Zariski open subset of $W^r \subset J^r$.

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*The order of Φ is the maximum value of k such that Φ explicitly depends on the variable $y^{(k)}$.

Signatures

Let $\mathcal{X}$ be a rational algebraic curve with a parameterization $(x(t), y(t))$.

- A restriction $\Phi|_{\mathcal{X}}$ of a rational differential function $\Phi(x, y, y^{(1)}, \dots, y^{(n)})$ to $\mathcal{X}$ is obtained by substituting:

$$y^{(1)} = \frac{\dot{y}}{\dot{x}} \quad , \dots , \quad y^{(k)} = \frac{\dot{y^{(k-1)}}}{\dot{x}}, \tag{3}$$

where $\dot{}$ denotes the derivative with respect to the parameter.*

- Let $\mathcal{I} = \{K, T\}$ be differentially separating set for G-action and let $\mathcal{X}$ be non-exceptional with respect to $\mathcal{I}$. The signature $\mathcal{S}_{\mathcal{X}}$ is the standard topology closure of the image of the rational map $S|_{\mathcal{X}}\colon \mathbb{R} \to \mathbb{R}^2$ defined by $S|_{\mathcal{X}}(t) = (K|_{\mathcal{X}}(t)\,,\, T|_{\mathcal{X}}(t))$.

- Theorem. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are non-exceptional with respect to $\mathcal{I}$, then
$$\boxed{\mathcal{X}_1 \cong_G \mathcal{X}_2 \quad \Longleftrightarrow \quad \mathcal{S}_{\mathcal{X}_1} = \mathcal{S}_{\mathcal{X}_2}.}$$

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*If $\mathcal{X}$ is not a vertical line and the denominator if Φ is not annihilated by substitution (3), then $\Phi|_{\mathcal{X}}$ is a rational function $\mathbb{R} \to \mathbb{R}$, otherwise it is undefined.

Differentially separating sets for $\mathcal{A}(2)$ and $\mathcal{PGL}(3)$ actions.

Classical G-curvatures and G-arclengths:

$SE(2)$: $\kappa = \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}, \quad ds = \sqrt{\dot{x}^2+\dot{y}^2}\,dt \;\Rightarrow \kappa_s = \frac{d\kappa}{ds},\, \kappa_{ss}, \ldots$

$\mathcal{SA}(2)$: $\mu = \frac{3\,\kappa\,(\kappa_{ss}+3\,\kappa^3)-5\,\kappa_s^2}{9\,\kappa^{8/3}}, \quad d\alpha = \kappa^{1/3} ds \;\Rightarrow\; \mu_\alpha = \frac{d\mu}{d\alpha},\; \mu_{\alpha\alpha},\; \ldots$

$\mathcal{PGL}(3)$: $\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha - 7\,\mu_{\alpha\alpha}^2 - 9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}, \quad d\rho = \mu_\alpha^{1/3} d\alpha \;\Rightarrow\; \eta_\rho = \frac{d\eta}{d\rho}, \ldots.$

Theorem

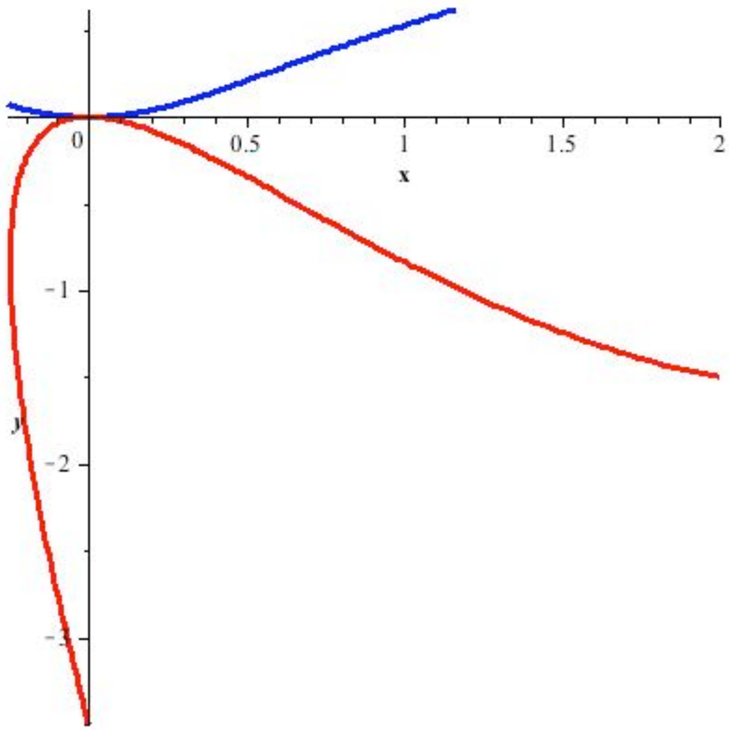
- $\mathcal{I}_{\mathcal{A}} = \left\{K_a = \frac{(\mu_\alpha)^2}{\mu^3}, \quad T_a = \frac{\mu_{\alpha\alpha}}{3\,\mu^2}\right\}$ is a differentially separating set of $\mathcal{A}(2)$ rational invariants, whose exceptional curves are lines and parabolas.

- $\mathcal{I}_{\mathcal{PGL}} = \left\{K_p = \eta^3, \quad T_p = \eta_\rho\right\}$ is a differentially separating set of $\mathcal{PGL}(3)$ rational invariants, whose exceptional curves are lines and conics.

Examples of $\mathcal{A}(2)$-signatures

Is $\alpha(t) = (t^4 + t,\, t^2)$ implicitly defined by $-y + x^2 - 2\,y^2\,x + y^4 = 0$

$\mathcal{A}(2)$-equivalent to $\beta(s) = (s^2 + s,\, s^3 - 3\,s^2 + \frac{1}{2}\,s^4)$ implicitly defined by $14\,y + 42\,x^2 + 28\,y\,x + 4\,y^2 - 14\,x^3 - 4\,x^2\,y + x^4 = 0$?



- The signature $\mathcal{S}_\alpha$ for $\alpha(t) = (t^4 + t,\ t^2)$ is a parametric curve

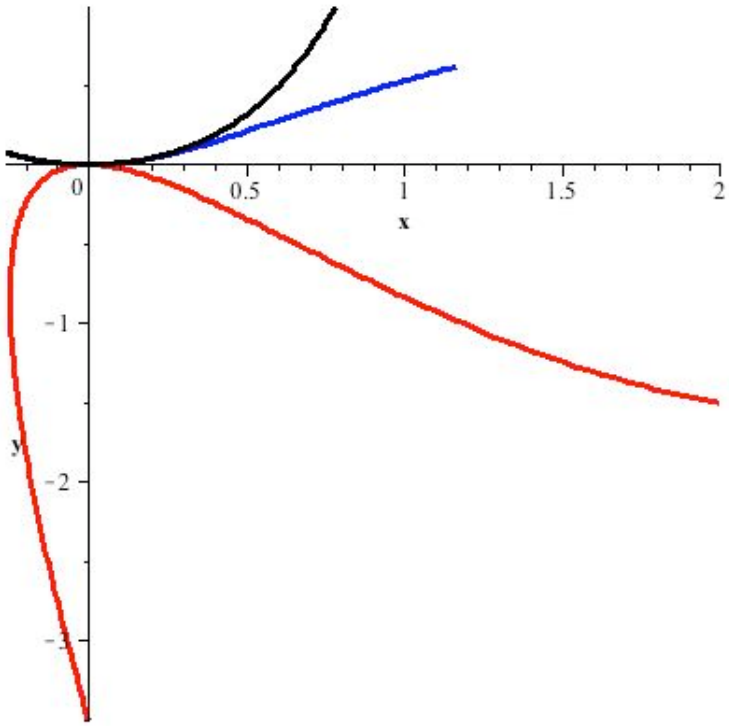
$$K_a|_\gamma(t) = -\frac{1}{40}\,\frac{(56\,t^3+1)^2}{t^3}, \quad T_a|_\gamma(t) = -\frac{168}{5}\,t^3 - \frac{11}{5}.$$

- The signature $\mathcal{S}_\beta$ for $\beta(s) = (s^2 + s,\ s^3 - 3\,s^2 + \frac{1}{2}\,s^4)$ is a parametric curve

$$K_a|_\beta(s) = -\frac{28}{5}\,\frac{(4\,s^3+6\,s^2+3\,s+1)^2}{(2\,s+1)^3}, \quad T_a|_\beta(s) = -\frac{2}{5}(12\,s^3+18\,s^2+9\,s-7).$$

- Is it true that $\mathcal{S}_\alpha = \mathcal{S}_\beta$ and hence α and β are $\mathcal{A}(2)$-equivalent?
  - $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ have the same implicit equation: $\boxed{K_a\,(165+75\,T_a) - 175\,T_a^2 - 560\,T_a - 448 = 0.}$ Over $\mathbb{C}$ it is a sufficient condition, but not over $\mathbb{R}$.
  - We can look for a real reparameterization by solving $T_a|_\alpha(t) = T_a|_\beta(s)$ for t in terms of s: $t = \frac{1}{14}\,7^{2/3}\,(2\,s+1)$ indeed works. Yes!!!

Is $\gamma(t) = (t, t^4 + t^2)$ $\mathcal{A}(2)$-equivalent to α and β?



No! because its signature $\mathcal{S}_\gamma$ has a different implicit equation:

$$\boxed{40000 - 6000\,T_a + 14525\,K_a - 1575\,T_a^2 + 3780\,K_a\,T_a - 448\,K_a^2 + 245\,T_a^3 = 0.}$$

Algorithm for central projections.

INPUT: Parameterizations $\Gamma = \big(z_1, z_2, z_3) \in \mathbb{Q}(s)^3$ and $\gamma = (x,\, y) \in \mathbb{Q}(t)^2$ of rational algebraic curves $\mathcal{Z} \subset \mathbb{R}^3$ and $\mathcal{X} \subset \mathbb{R}^2$, such that $\dot{\Gamma} \times \ddot{\Gamma} \neq 0$. *

OUTPUT: The truth of the statement:

$$\exists [P] \in \mathcal{CP}, \text{ such that } \mathcal{X} = P(\mathcal{Z}).$$

NON-RIGOROUS OUTLINE:

1. if $\mathcal{X}$ is a line then $\mathcal{Z}$ can be projected to $\mathcal{X}$ is and only if
2. $\epsilon := \left(\frac{z_1+c_1}{z_3+c_3}, \frac{z_2+c_2}{z_3+c_3}\right)$ is a family of parametric curves.
3. if $\mathcal{X}$ is a conic then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c_1, c_2, c_3$ such that $\epsilon(s)$ is a conic.
4. if $\mathcal{X}$ is neither a line or a conic then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c_1, c_2, c_3$ such that signature $\epsilon(s)$ belongs to the signature of $\gamma(t)$.

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*Equivalently, $\mathcal{Z}$ is not a line.

Differentially separating set of rational $\mathcal{PGL}(3)$-invariants:

$$\boxed{\Delta_2 = 9\, y^{(5)}\, [y^{(2)}]^2 - 45\, y^{(4)}\, y^{(3)}\, y^{(2)} + 40\, [y^{(3)}]^3.}$$

$$\begin{aligned}
K_{\mathcal{P}} \;&=\; \frac{729}{8\,(\Delta_2)^8}\Big(18\, y^{(7)}\, [y^{(2)}]^4\, \Delta_2 - 189\, [y^{(6)}]^2\, [y^{(2)}]^6 \\
&+\; 126\, y^{(6)}\, [y^{(2)}]^4\, (9\, y^{(5)}\, y^{(3)}\, y^{(2)} + 15\, [y^{(4)}]^2\, y^{(2)} - 25\, y^{(4)}\, [y^{(3)}]^2) \\
&-\; 189\, [y^{(5)}]^2\, [y^{(2)}]^4\, (4\, [y^{(3)}]^2 + 15\, y^{(2)}\, y^{(4)}) \\
&+\; 210\, y^{(5)}\, y^{(3)}\, [y^{(2)}]^2\, (63\, [y^{(4)}]^2\, [y^{(2)}]^2 - 60\, y^{(4)}\, [y^{(3)}]^2\, y^{(2)} + 32\, [y^{(3)}]^4) \\
&-\; 525\, y^{(4)} y^{(2)}\, (9\, [y^{(4)}]^3\, [y^{(2)}]^3 + 15\, [y^{(4)}]^2\, [y^{(3)}]^2\, [y^{(2)}]^2 - 60\, y^{(4)}\, [y^{(3)}]^4\, y^{(2)} + 64\, [y^{(3 \\
&+\; 11200\, [y^{(3)}]^8\Big)^3;
\end{aligned}$$

$$\begin{aligned}
T_{\mathcal{P}} \;&=\; \frac{243\, [y^{(2)}]^4}{2\,(\Delta_2)^4}\Big(2\, y^{(8)}\, y^{(2)}\, (\Delta_2)^2 \\
&-\; 8\, y^{(7)}\, \Delta_2\, (9\, y^{(6)}\, [y^{(2)}]^3 - 36\, y^{(5)}\, y^{(3)}\, [y^{(2)}]^2 - 45\, [y^{(4)}]^2\, [y^{(2)}]^2 + 120\, y^{(4)}\, [y^{(3)}]^2\, y^{(2)} \\
&+\; 504\, [y^{(6)}]^3\, [y^{(2)}]^5 - 504\, [y^{(6)}]^2\, [y^{(2)}]^3\, (9\, y^{(5)}\, y^{(3)}\, y^{(2)} + 15\, [y^{(4)}]^2\, y^{(2)} - 25\, y^{(4)}\, [y^{(3)} \\
&+\; 28\, y^{(6)}\, \big(432\, [y^{(5)}]^2\, [y^{(3)}]^2\, [y^{(2)}]^3 + 243\, [y^{(5)}]^2\, y^{(4)}\, [y^{(2)}]^4 - 1800\, y^{(5)}\, y^{(4)}\, [y^{(3)}]^3\, [y^{(2 \\
&-\; 240 y^{(5)}\, [y^{(3)}]^5\, y^{(2)} + 540 y^{(5)}\, [y^{(4)}]^2\, [y^{(3)}]\, [y^{(2)}]^3 + 6600\, [y^{(4)}]^2\, [y^{(3)}]^4\, y^{(2)} - 2000 y \\
&-\; 5175\, [y^{(4)}]^3\, [y^{(3)}]^2\, [y^{(2)}]^2 + 1350\, [y^{(4)}]^4\, [y^{(2)}]^3\big) - 2835\, [y^{(5)}]^4\, [y^{(2)}]^4 \\
&+\; 252\, [y^{(5)}]^3 y^{(3)}\, [y^{(2)}]^2\, (9 y^{(4)}\, y^{(2)} - 136\, [y^{(3)}]^2) - 35840\, [y^{(5)}]^2\, [y^{(3)}]^6 \\
&-\; 630\, [y^{(5)}]^2\, [y^{(4)}]\, [y^{(2)}]\, (69\, [y^{(4)}]^2\, [y^{(2)}]^2 - 160\, [y^{(3)}]^4 - 153\, y^{(4)}\, [y^{(3)}]^2\, [y^{(2)}]) \\
&+\; 2100\, y^{(5)}\, [y^{(4)}]^2\, y^{(3)}\, (72\, [y^{(3)}]^4 + 63\, [y^{(4)}]^2\, [y^{(2)}]^2 - 193\, y^{(4)}\, [y^{(3)}]^2\, y^{(2)}) \\
&-\; 7875\, [y^{(4)}]^4\, (8\, [y^{(4)}]^2\, [y^{(2)}]^2 - 22 y^{(4)}\, [y^{(3)}]^2\, [y^{(2)}] + 9\, [y^{(3)}]^4)\Big).
\end{aligned}$$

The restriction of $K_{\mathcal{P}}|_{\mathcal{X}}$ and $T_{\mathcal{P}}|_{\mathcal{X}}$ to a planar curve $\mathcal{X}$ with rational parameterization $(x(t), y(t))$ is computed by substitution

$$y^{(1)} = \frac{\dot{y}}{\dot{x}} \quad, \ldots, \quad y^{(k)} = \frac{\dot{y^{(k-1)}}}{\dot{x}}, \tag{5}$$

into the formulas for invariants.

- $y^{(1)}, \ldots, y^{(k)}$ are rational functions of t unless $\mathcal{X}$ is a vertical line.
- Invariants $K_{\mathcal{P}}|_{\mathcal{X}}$ and $T_{\mathcal{P}}|_{\mathcal{X}}$ are rational functions of t unless $\Delta_2|_{\mathcal{X}} \underset{\mathbb{R}(t)}{=} 0$.
- $\Delta_2|_{\mathcal{X}} \underset{\mathbb{R}(t)}{=} 0$ if and only if $\mathcal{X}$ is a line or a conic.
- When the restriction of invariants to the family of curves $\tilde{\mathcal{Z}}_c$ parametrized by $\epsilon(c, s) := \left(\frac{z_1(s)+c_1}{z_3(s)+c_3}, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$ is computed the differentiation in (5) is taken with respect to s.
- For the values c, such that $\epsilon(c, s)$ is not a line or a conic, specialization of c commutes with restriction of invariants $K_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}$ and $T_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}$.

ALGORITHM:

1. if $\left|\begin{array}{c}\dot{\gamma}\\ \ddot{\gamma}\end{array}\right| \underset{\mathbb{R}(t)}{=} 0$ then if $\left|\begin{array}{c}\dot{\Gamma}\\ \ddot{\Gamma}\\ \dddot{\Gamma}\end{array}\right| \underset{\mathbb{R}(s)}{=} 0$ then return TRUE else return FALSE;
2. $\epsilon := \left(\frac{z_1+c_1}{z_3+c_3}, \frac{z_2+c_2}{z_3+c_3}\right) \in \mathbb{Q}(c_1, c_2, c_3, s)^2$;
3. if $\Delta_2|_\gamma \underset{\mathbb{R}(t)}{=} 0$ then if $\exists (c_1, c_2, c_3) \in \mathbb{R}^3$

$$z_3 + c_3 \underset{\mathbb{R}(s)}{\neq} 0 \wedge \left|\begin{array}{c}\dot{\epsilon}\\ \ddot{\epsilon}\end{array}\right| \underset{\mathbb{R}(s)}{\neq} 0 \wedge \Delta_2|_\epsilon \underset{\mathbb{R}(s)}{=} 0$$

   then return TRUE else return FALSE.
4. return the truth of the statement:

$$\exists\, (c_1, c_2, c_3) \in \mathbb{R}^3$$

$$z_3 + c_3 \underset{\mathbb{R}(s)}{\neq} 0 \wedge \left|\begin{array}{c}\dot{\epsilon}\\ \ddot{\epsilon}\end{array}\right| \underset{\mathbb{R}(s)}{\neq} 0 \wedge \Delta_2|_\epsilon \underset{\mathbb{R}(s)}{\neq} 0 \tag{6}$$

$$\wedge \forall s \in \mathbb{R}$$

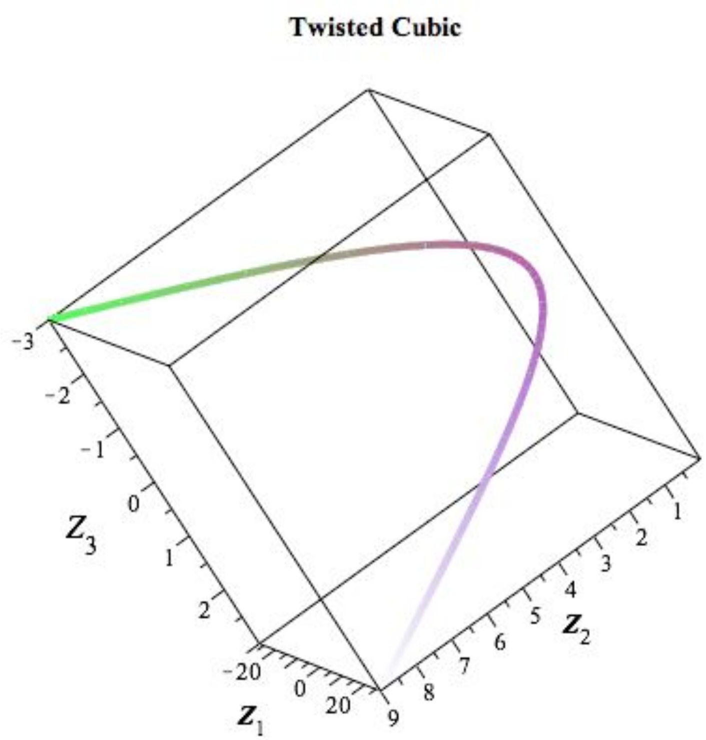
$$\Delta_2|_\epsilon \underset{\mathbb{R}}{\neq} 0 \;\Rightarrow\; \exists t \in \mathbb{R}$$

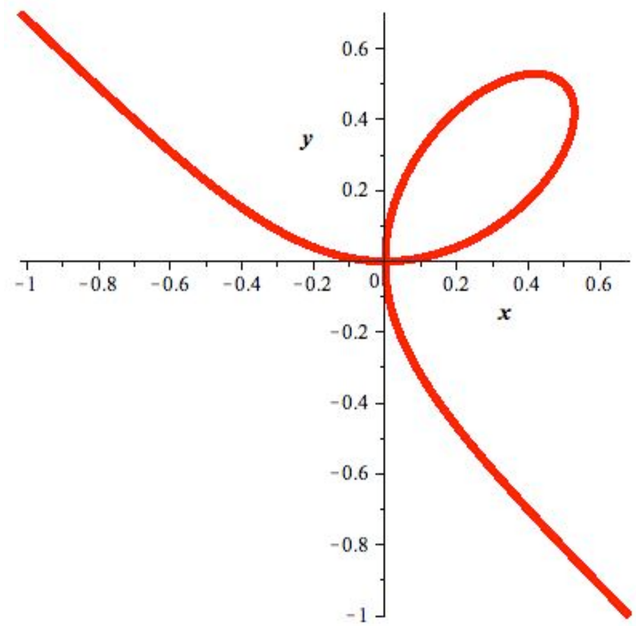
$$K_{\mathcal{P}}|_\epsilon \underset{\mathbb{R}}{=} K_{\mathcal{P}}|_\gamma \wedge T_{\mathcal{P}}|_\epsilon \underset{\mathbb{R}}{=} T_{\mathcal{P}}|_\gamma.$$

Example: central projections of the twisted cubic

Can the twisted cubic $\mathcal{Z}$ parametrized by

$$\Gamma(s) = \left(s^3,\, s^2,\, s\right) \, , \, s \in \mathbb{R}$$





be projected to a curve $\mathcal{X}_1$ parametrized by $\alpha(t) = \left(\frac{t}{t^3+1}, \frac{t^2}{t^3+1}\right)$ with an implicit equation $x^3 + y^3 - yx = 0$?

- Since $\mathcal{X}_1$ is not a line or a conic its signature is defined and is parametrized by invariants:

$$K_{\mathcal{P}}|_{\gamma_3}(t) = -\frac{9261}{50}\frac{t^7 - t^4 + t}{(t^3-1)^8}, \; T_{\mathcal{P}}|_{\gamma_3}(t) = -\frac{21}{10}\frac{(t^3+1)^4}{(t^3-1)^4}.$$

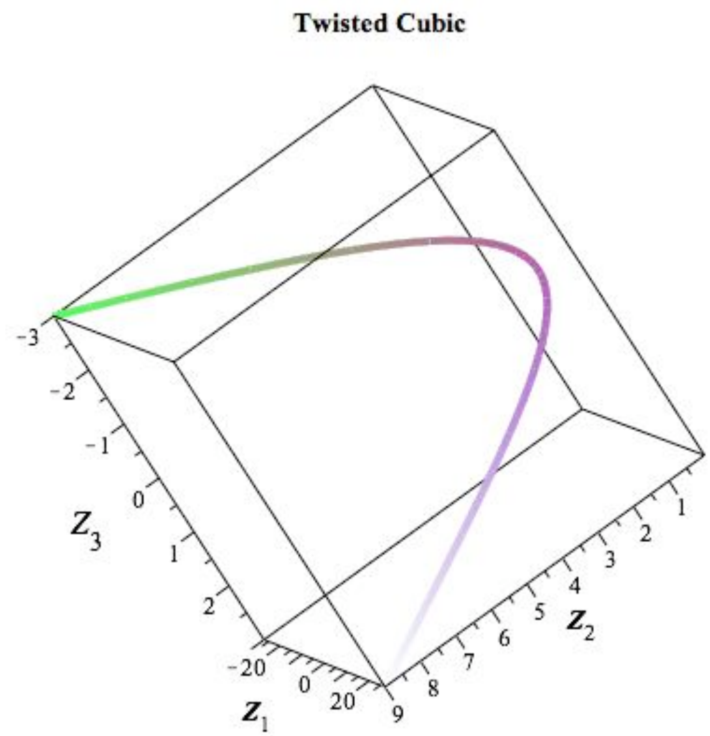
- We need to determine if there exists c such that a curve parametrized by $\epsilon(c_1, c_2, c_3, s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$ is not a line or a conic and has the same signature as $\mathcal{X}_1$.

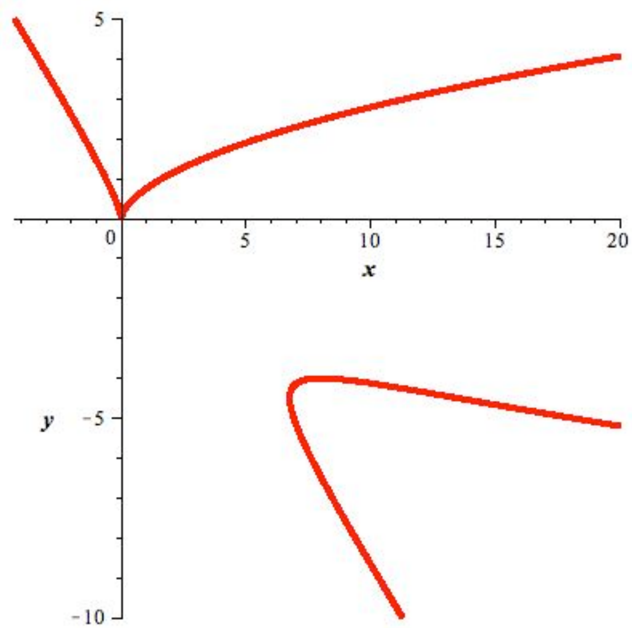
- This is indeed true for c=(1,0,0).

- Yes!! The twisted cubic can be projected to $x^3 + y^3 - yx = 0$. A possible projection is $x = \frac{z_3}{z_1+1}$, $y = \frac{z_2}{z_1+1}$.

Can the twisted cubic $\mathcal{Z}$ parametrized by

$$\Gamma(s) = \left(s^3,\, s^2,\, s\right)\ ,\ s \in \mathbb{R}$$





be projected to a curve $\mathcal{X}_3$ parametrized by $\beta(t) = \left(\frac{t^3}{t+1}, \frac{t^2}{t+1}\right)$ with an implicit equation $y^3 + y^2 x - x^2 = 0$?

- Since $\mathcal{X}_2$ is not a line or a conic its signature is defined and parametrized by invariants:

$$K_{\mathcal{P}}|_{\gamma_2}(t) = \frac{250047}{12800} \text{ and } T_{\mathcal{P}}|_{\gamma_2}(t) = 0, \quad \forall t \in \mathbb{R}.$$

- We need to determine if there exists c such that a curve parametrized by $\epsilon(c_1, c_2, c_3, s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$ is not a line or a conic and has the same constant invariants as $\mathcal{X}_2$.

- This is indeed true for c=(0,0,1).

- Yes!! The twisted cubic can be projected to $y^3 + y^2 x - x^2 = 0$. A possible projection is $x = \frac{z_1}{z_3+1}$, $y = \frac{z_2}{z_3+1}$.

Can the twisted cubic be projected to quadric $\mathcal{X}_3$ parameterized by $\gamma = (t^2, t)$?

- Does there exists c such that a curve parametrized by $\epsilon(c_1, c_2, c_3, s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$ is a quadric, i.e $\Delta_2|_\epsilon = 0$?
- Yes!! $c_1 = c_2 = c_3 = 0$

Can the twisted cubic be projected to quintic $\mathcal{X}_4$ parameterized by $\delta = (t, t^5)$?

- The signature of $\mathcal{X}_4$ degenerates to a point:

$$K_{\mathcal{P}}|_{\gamma_4}(t) = \frac{1029}{128} \text{ and } T_{\mathcal{P}}|_{\gamma_4}(t) = 0\,, \quad \forall t.$$

- Does there exists c such that a curve parametrized by $\epsilon(c_1, c_2, c_3, s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$ is not a line or a conic and

$$K_{\mathcal{P}}|_\epsilon(c, s) = \frac{1029}{128} \text{ and } T_{\mathcal{P}}|_\epsilon(c, s) = 0,\ \forall s \in \mathbb{R}?$$

- NO!! Substitution of several values of s gives an inconsistent system on c.

In the above example, although $\mathcal{Z}$ can be projected to each of the planar $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ none of the planar curves are $\mathcal{PGL}(3)$-equivalent.

Parallel projection example: $\Gamma(s) = \left(s^4 + 1,\, s^2,\, s\right)$

projects to

$$\gamma_1(t) = \left(t^4 + t\,,\, t^2\right), \text{ with } (i,j,k) = (1,2,3) \text{ and } c_1 = 0, c_2 = \frac{1}{2}$$

and to

$$\gamma_2(t) = \left(t^3 - t\,,\, t^3 + t^2\right) \text{ with } (i,j,k) = (1,2,3) \text{ and } c_1 = c_2 = 0$$

but not to $\gamma_3(t) = \left(t/(1+t^3),\, t^2/(1+t^3)\right).$

Remark: $\gamma_1(t)$ and $\gamma_2(t)$ are not $\mathcal{A}(2)$-equivalent

Can we use the same method to solve the projection problem for non-rational curves? In principle, yes, but

one has to be careful when describing a family of planar curves

$$\tilde{\mathcal{Z}}_c = \overline{\left\{ \left(\frac{z_1 + c_1}{z_3 + c_3}, \frac{z_2 + c_2}{z_3 + c_3} \right) \middle| (z_1, z_2, z_3) \in \mathcal{Z} \right\}}$$

by an implicit equation. Let an irreducible algebraic curve $\mathcal{Z} \subset \mathbb{C}^3$ be a zero set of a prime ideal Z and

$$\begin{aligned} A &= Z + \left\langle x\,(z_3 + c_3) - (z_1 + c_1),\ y\,(z_3 + c_3) - (z_2 + c_2),\ \delta\,(z_3 + c_3) - 1 \right\rangle \\ &\subset \mathbb{C}[c, x, y, z_1, z_2, z_3, \delta]. \end{aligned}$$

Unfortunately, in general, elimination does not commute with specialization:

We can substitute a value c^* into A and then $B^* = A^* \cap \mathbb{C}[x, y]$ is an ideal of $\tilde{\mathcal{Z}}_{c^*}$, but if we first compute $B = A \cap \mathbb{C}[c, x, y]$ and then substitute c^*, we might get a different answer.

Example (twisted cubic)

For $Z = < z_1 - z_2\, z_3, \quad z_2 - z_3^2, \quad z_1\, z_3 - z_2^2 >$

$$\tilde{\mathcal{Z}}_c = \overline{\left\{ \left(\frac{z_1 + c_1}{z_3 + c_3}, \frac{z_2 + c_2}{z_3 + c_3} \right) \middle| \, (z_1, z_2, z_3) \in \mathcal{Z} \right\}}$$

is defined by

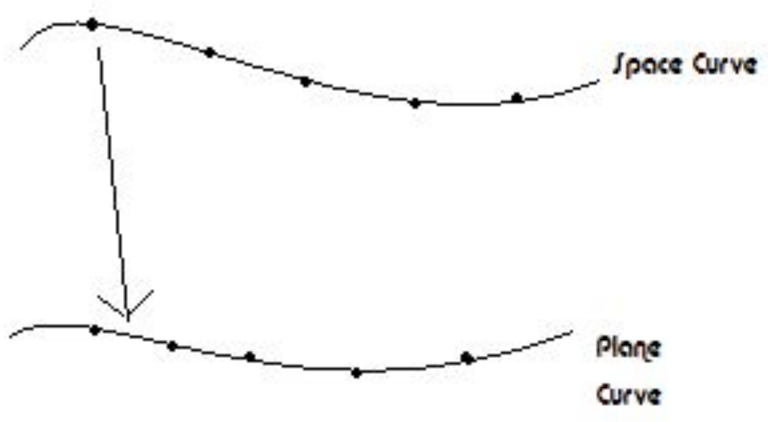
$$\begin{aligned} 0 \;=\; & (-c_3^2 - c_2)\, x^2 + (c_3^2 + c_2)\, y^2\, x + (c_1 + c_3\, c_2)\, x\, y + \\ & (2\, c_1\, c_3 - 2\, c_2^2)\, x + (c_3^3 - c_1)\, y^3 + (-3\, c_1\, c_3 - 3\, c_3^2\, c_2)\, y^2 + \\ & (3\, c_2^2\, c_3 + 3\, c_1\, c_2)\, y - c_1^2 - c_2^3 \end{aligned}$$

unless c is in the zero set of $< c_3^2 + c_2, \quad c_1 - c_3^3 >$.

Then $\tilde{\mathcal{Z}}_c$ is defined by $y^2 - x + c_3\, y + c_3^2 = 0$.

Continuous vs. discrete:

Projection problem for curves vs. projection problems for finite lists of points.



If $\mathbf{Z} = (\mathrm{z}^1, \ldots, \mathrm{z}^m)$ is a discrete sampling of a curve $\mathcal{Z}$ and $\mathbf{X} = (\mathrm{x}^1, \ldots, \mathrm{x}^m)$ is a discrete sampling of $\mathcal{X}$, these sets might not be in a correspondence under a projection even when the curves $\mathcal{Z}$ and $\mathcal{X}$ are related by a projection.

Projection criteria for list of points*:

($\mathcal{CP}$) A list $\mathbf{Z} = (\mathbf{z}^1, \ldots, \mathbf{z}^m)$ of m points with coordinates $\mathbf{z}^i = (z_1^r, z_2^r, z_3^r)$, $r = 1 \ldots m$, projects onto a list $X = (\mathbf{x}^1, \ldots, \mathbf{x}^m)$ of m points in $\mathbb{R}^2$ with coordinates $\mathbf{x}^r = (x^r, y^r)$ by a finite projection if and only if there exist $c_1, c_2, c_3 \in \mathbb{R}$ and $[A] \in \mathcal{PGL}(3)$, such that

$$[x^r, y^r, 1]^T = [A][z_1^r + c_1,\ z_2^r + c_2,\ z_3^r + c_3]^T \text{ for } r = 1 \ldots m.$$

($\mathcal{PP}$) A list $\mathbf{Z} = (\mathbf{z}^1, \ldots, \mathbf{z}^m)$ of m points in $\mathbb{R}^3$ with coordinates $\mathbf{z}^i = (z_1^r, z_2^r, z_3^r)$, $r = 1 \ldots m$, projects onto a list $X = (\mathbf{x}^1, \ldots, \mathbf{x}^m)$ of m points in $\mathbb{R}^2$ with coordinates $\mathbf{x}^r = (x^r, y^r)$ by an affine projection if and only if there exist $c_1, c_2 \in \mathbb{R}$, an ordered triplet $(i, j, k) \in \{(1,2,3),\ (1,3,2),\ (2,3,1)\}$ and $[A] \in \mathcal{A}(2)$, such that

$$[x^r, y^r, 1]^T = [A]\left[z_i^r + c_1\, z_k^r,\ z_j^r + c_2\, z_k^r, 1\right]^T \text{ for } r = 1 \ldots m.$$

---

*separating sets of algebraic invariants can be used to solve group-equivalence problems for sets of points

More details

- Maple Code `http://www.math.ncsu.edu/~iakogan/symbolic/projections.html`

- Burdis, J. and Kogan I. “Object image correspondence for curves under parallel and central projections”, 10 pp, accepted to the Symposium on Computational Geometry, SoCG 2012.

- `http://arxiv.org/abs/1202.1303` (implicit case needs updating)

- Burdis, J. “Object Image correspondance under projections”, 2010, Ph. D. Thesis, NCSU.

Thank you !!!