Invariant Calculus with VESSIOT

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- *Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex*, I. Kogan, P.J.Olver.
- MAPLE *package* VESSIOT *for calculus on the jet bundles.* Utah State University, I.Anderson et al.

Group action on jet bundles

$$\boxed{G \curvearrowright M \Rightarrow G \curvearrowright \{p\text{-dimensional submanifolds of } M\} \Rightarrow G \curvearrowright J^k(M,p)}$$

Jet spaces: $J^k = J^k(M,p)$ are bundles over M.
The fiber over $z \in M$ consists of the equivalence classes of p-dim. submanifolds with k-th order contact at z.

Local coordinates on J^k:
$\mathbf{x} = (x^1, \ldots, x^p)$ – independent variables,
$\mathbf{u} = (u^1, \ldots u^q)$ – dependent variables,
$\mathbf{u}^{(\mathbf{k})} = (u^\alpha_J,\ \alpha = 1 \ldots q,\ J = \text{multi-index}, |J| \le k)$ – "derivatives"

$$J = (j_1, \ldots, j_p) \qquad 0 \le j_i, \qquad |J| = j_1 + \cdots + j_p, \tag{1}$$

$$J = (j_1, \ldots, j_k)^S, \quad 1 \le j_i \le p, \quad |J| = k \tag{2}$$

Projections: $\pi^n_k : J^n \longrightarrow J^k$ for $k \leq n$:

$$J^\infty \cdots \longrightarrow J^k \longrightarrow J^{k-1} \longrightarrow \ldots J^1 \longrightarrow J^0 = M$$

Jets of submanifolds:

If $N : u^\alpha = f^\alpha(\mathbf{x})$ then $j^k(N) : u^\alpha = f^\alpha(\mathbf{x})$, $\qquad u^\alpha_J = \frac{\partial^k f^\alpha}{\partial^{j_1} x^1 \ldots \partial^{j_p} x^p}$.

Prolongation of the action:

$$g \cdot j^k(N) = j^k(g \cdot N).$$

Contact structure on $J^\infty(M,p)$

Basis of horizontal sub-bundle	Basis of vertical sub-bundle

Tangent

total derivatives: $\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} u_i^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\alpha,J} u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}$	vertical derivatives $\frac{\partial}{\partial u^\alpha}$, $\frac{\partial}{\partial u_J^\alpha}$, $\alpha = 1 \ldots q$

Cotangent

horizontal one-forms $d\,x^1, \ldots, d\,x^p$	contact one-forms $\theta^\alpha = du^\alpha - \sum_{i=1}^{p} u_i^\alpha\, dx^i$, $\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{Ji}^\alpha\, dx^i$.

Bigrading of exterior differential algebra:

$$\text{Grading: } \Lambda^* = \bigoplus \Lambda^k, \text{ where } \Lambda^k = \left\{ \sum \underbrace{\text{one form} \wedge \cdots \wedge \text{one form}}_{k \text{ times}} \right\}.$$

$$d\colon \Lambda^k \longrightarrow \Lambda^{k+1}, \quad d \circ d = 0 \Longrightarrow \text{ de Rham complex.}$$

$$\text{Bigrading: } \Lambda^* = \bigoplus \Lambda^{s,t}, \text{ where } \Lambda^{s,t} =$$

$$\left\{ \sum \underbrace{\text{hor. 1-form} \wedge \cdots \wedge \text{hor. 1-form}}_{s \text{ times}} \wedge \underbrace{\text{cont. 1-form} \wedge \cdots \wedge \text{cont. 1-form}}_{t \text{ times}} \right\}$$

$$d\colon \Lambda^{s,t} \longrightarrow \Lambda^{s+1,t} \oplus \Lambda^{s,t+1} \Rightarrow d = d_H + d_V$$

$$d^2 = (d_H + d_V)^2 = 0 \Rightarrow d_H^2 = 0,\ d_V^2 = 0,\ d_H \circ d_V = -d_V \circ d_H$$

$$\Downarrow$$

Bicomplex

Variational Bicomplex

Tulczyjew, Tsujishita, Vinogradov, Anderson

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & \uparrow\partial_V \\
\Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,2} & \xrightarrow{d_H} & \Lambda^{p,2} & \xrightarrow{I} & \mathcal{F}^2 \\
d_V\uparrow & & d_V\uparrow & & \uparrow & & d_V\uparrow & & d_V\uparrow & & \uparrow\partial_V \\
\Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,1} & \xrightarrow{d_H} & \Lambda^{p,1} & \xrightarrow[I]{} & \mathcal{F}^1 \\
d_V\uparrow & & d_V\uparrow & & \uparrow & & d_V\uparrow & & \uparrow & \nearrow\partial_V & \\
\Lambda^{0,0} & \xrightarrow[d_H]{} & \Lambda^{1,0} & \xrightarrow[d_H]{} & \dots & \xrightarrow[d_H]{} & \Lambda^{p-1,0} & \xrightarrow[d_H]{} & \Lambda^{p,0} & &
\end{array}
$$

$I\colon \Lambda^{p,s} \longrightarrow \mathcal{F}^s = \Lambda^{p,s}/Im\, d_H$ - integration by parts operator

$\partial_v = I \circ d_V$- variational derivative.

Integration by parts operator, $\lambda \in \Lambda^{p,s}$:

$$I(\lambda) = \sum_\alpha \frac{1}{s}\theta^\alpha \wedge \left(\left\langle \lambda; \frac{\partial}{\partial u^\alpha} \right\rangle + \sum_J \left(-\frac{d}{dx_J} \right) \left\langle \lambda, \frac{\partial}{\partial u_J^\alpha} \right\rangle \right).$$

$\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \xrightarrow{d_V} \sum_{\alpha,J} \frac{\partial L}{\partial u_J^\alpha} \theta_J^\alpha \wedge d\mathbf{x} \quad \xrightarrow{I} \sum_{\alpha,J} E^\alpha(L) \theta^\alpha \wedge d\mathbf{x}$

$E^\alpha(L) = 0, \quad \alpha = 1, \ldots, q$ – Euler-Lagrange equations.

$\Lambda^{p,0}$–Lagrangians $\supset d_H \Lambda^{p-1,0} = (\mathrm{loc}) = \ker \partial_V$–trivial Lagrangians.

$\mathcal{F}^1$–source forms $\supset \partial_V \Lambda^{p,0} = (\mathrm{loc}) = \ker \partial_V$–Euler-Lagrange forms.

Invariant structure on $J^\infty(M,p)$.

Invariant horizontal basis	Invariant vertical basis

Tangent

invariant total diff. operators $\tilde{\mathcal{D}}_1, \ldots \tilde{\mathcal{D}}_p$ $span\left\{\tilde{\mathcal{D}}_i\right\} = span\left\{\frac{d}{dx_i}\right\}$	invariant vertical diff. operators $\tilde{\mathcal{C}}^\alpha, \tilde{\mathcal{C}}^\alpha_J$ $span\left\{\tilde{\mathcal{C}}^\alpha\right\} \neq span\left\{\frac{\partial}{\partial u^\alpha}\right\}$ unless the action is projectable

Cotangent

invariant "horizontal" one-forms $\tilde{\omega}^1, \ldots, \tilde{\omega}^p$ $\tilde{H} = span\left\{\tilde{\omega}^i\right\} \neq H = span\left\{d\,x^i\right\}$ unless the action is projectable	invariant contact one-forms $\tilde{\theta}^\alpha, \tilde{\theta}^\alpha_J$ $span\left\{\tilde{\theta}^\alpha, \tilde{\theta}^\alpha_J\right\} = span\left\{\theta^\alpha, \theta^\alpha_J\right\}$

A generating set of differential invariants.

Thm. (Tresse) $\exists\{I^1, \ldots, I^n\}$ - invariant functions on $J^\infty(M,p)$, s. t. any invariant function

$$I = F\left(\ldots, \tilde{\mathcal{D}}_J(I^l), \ldots\right)$$

$$\Downarrow$$

non-free differential algebra (with syzygies):

$$S\left(\ldots, \tilde{\mathcal{D}}_J(I^l), \ldots\right) = 0$$

Problems:

- Finding a (minimal) set of generating invariants.
- Finding a (minimal) set of generating syzygies.
- Finding invariant basis for differential forms.

Solution: invariantization via moving frame method (Olver, Fels).

Euclidean group $SE(2) = SO(2) \ltimes R^2$ acts on plane curves $u = u(x)$:

$$\boxed{x \mapsto \cos(\phi)x - \sin(\phi)u + a, \quad u \mapsto \sin(\phi)x + \cos(\phi)u + b.}$$

- a generating set: $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$, $\tilde{\mathcal{D}} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\frac{d}{dx}$,
- any invariant $I = F(\kappa, \kappa_s, \kappa_{ss}, \dots)$, no syzygies,
- $ds = \sqrt{1+u_x^2}d\,x$ is not an invariant form:
  $g^* ds = ds +$ contact form.
- invariant "horizontal" form: $\tilde{\omega} = ds + \frac{u_x}{\sqrt{1+u_x^2}}\theta$.
- invariant contact forms:
  $\tilde{\theta} = \frac{\theta}{\sqrt{1+u_x^2}}, \quad \tilde{\theta}_1 = \frac{(1+u_x^2)\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}, \dots$

$SE(2) = SO(2) \ltimes R^2$ acts on surfaces $u = u(x,y)$ in $\mathbb{R}^3$:

$$\boxed{x \mapsto \cos(\phi)x - \sin(\phi)y + a, \quad y \mapsto \sin(\phi)x + \cos(\phi)y + b, \quad u \mapsto u}$$

- a minimal generating set: $I = u, \quad I_{xx} = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2}$, $\tilde{\mathcal{D}}_1 = \frac{1}{\sqrt{u_x^2+u_y^2}}\left(u_y \frac{d}{dx} - u_x \frac{d}{dy}\right), \quad \tilde{\mathcal{D}}_2 = \frac{1}{\sqrt{u_x^2+u_y^2}}\left(u_x \frac{d}{dx} + u_y \frac{d}{dy}\right).$

- generating syzygies:

$$\tilde{\mathcal{D}}_1 I = 0,$$

$$I_{xx}^2 - I_{xx}\tilde{\mathcal{D}}_2 I + 2\left(\tilde{\mathcal{D}}_1\tilde{\mathcal{D}}_2 I\right)^2 - (\tilde{\mathcal{D}}_2 I)\left(\tilde{\mathcal{D}}_1^2\tilde{\mathcal{D}}_2 I - \tilde{\mathcal{D}}_2 I_{xx}\right) = 0,$$

$$\left(\tilde{\mathcal{D}}_1\tilde{\mathcal{D}}_2\, I\right)\left(I_{xx} + \tilde{\mathcal{D}}_2^2\, I\right) + \left(\tilde{\mathcal{D}}_2\, I\right)\left(\tilde{\mathcal{D}}_2\,\tilde{\mathcal{D}}_1\,\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1\tilde{\mathcal{D}}_2^2\right) I = 0$$

- $[\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2] = \frac{I_{xx}}{\tilde{\mathcal{D}}_2 I}\tilde{\mathcal{D}}_1 + \frac{\tilde{\mathcal{D}}_1\tilde{\mathcal{D}}_2 I}{\tilde{\mathcal{D}}_2 I}\tilde{\mathcal{D}}_2$

- invariant horizontal forms:

$$\tilde{\omega}_1 = \frac{1}{\sqrt{u_x^2+u_y^2}}\,(u_y d\,x - u_x d\,y)\,, \quad \tilde{\omega}_2 = \frac{1}{\sqrt{u_x^2+u_y^2}}\,(u_x\, d\,x + u_y\, d\,y).$$

Application: symmetry reduction.

Thm. (S.Lie)

- (almost) any symmetric system of differential equations can be written in terms of differential invariants.
- (almost) any symmetric variational problem can be written in terms of differential invariants and invariant differential forms.

Invariant Variational Problems:

$$\text{for } \forall g \in G\colon \mathcal{L}[u] = \int_\Omega L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} = \int_\Omega g^* \left[L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \right]$$

$$\mathcal{L}[u] = \int L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \qquad \Leftrightarrow \qquad \mathcal{L}[u] = \int \tilde{L}(\ldots, \tilde{\mathcal{D}}_J I^l, \ldots) \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_p$$

$$\Big\downarrow E^\alpha = \sum_J \left(-\frac{d}{dx_J}\right) \frac{\partial}{\partial u_J^\alpha} \qquad\qquad \Big\downarrow \quad ?$$

$$\begin{array}{ccc} \Delta^\alpha(\mathbf{x}, \mathbf{u}^{(n)}) = 0, & \Longleftrightarrow & \tilde{\Delta}^\alpha \left(\tilde{\mathcal{D}}_{J_1} I^1, \ldots, \tilde{\mathcal{D}}_{J_N} I^N \right) = 0, \\ \alpha = 1, \ldots, q. & & \alpha = 1, \ldots, N. \end{array}$$

problem and examples: P. Griffiths, I. Anderson;
general formula: P. Olver, I. Kogan; V. Itskov.

Euclidean invariant Lagrangian:

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2\,(1+u_x^2)^{5/2}}\, dx \qquad \Longleftrightarrow \qquad \int \tfrac{1}{2}\kappa^2\, ds$$

$$\Big\downarrow\; E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_x} \qquad\qquad \Big\downarrow \qquad ?$$

$$\Delta = 0 \qquad\qquad \Longleftrightarrow \qquad \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

$$\Delta = \frac{1}{2}\,\frac{30\,{u_2}^3\,{u_1}^2 - 5\,{u_2}^3 - 20\,u_2\,u_1\,u_3 - 20\,u_2\,{u_1}^3\,u_3 + 2\,u_4 + 4\,u_4\,{u_1}^2 + 2\,u_4\,{u_1}^4}{(1+{u_1}^2)^{(9/2)}}.$$

Invariantization via moving frames.

(Fels, Olver)

Thm. (Ovsyannikov) The action of G is loc. effective on open subsets

$$\Downarrow$$

$\exists n \leq \dim G$ (the order of stabilization), s. t. the prolonged action is loc. free on an open dense $\mathcal{V}^n \subset J^n$.

$$\Downarrow$$

- $\dim \mathcal{O}_{\mathbf{z}^{(n)}} = \dim G, \quad \forall \mathbf{z}^{(n)} \in \mathcal{V}^n$
- $\exists$ (local) cross-section $\mathcal{K}^n$:
  - $\texttt{codim}\,\mathcal{K}^n = \dim \mathcal{O}_{\mathbf{z}^{(n)}}$,
  - $\mathcal{K}^n$ is transversal to $\mathcal{O}_{\mathbf{z}^{(n)}}$ for $\forall \mathbf{z}^{(n)} \in \mathcal{V}^n$,
  - $\mathcal{K}^n \cap \mathcal{O}_{\mathbf{z}^{(n)}}$ consists of at most one point $\Longleftrightarrow$ regular action.

Define $\rho : \mathcal{V}^n \longrightarrow G$, by the condition $\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)} \in \mathcal{K}^n$

$$\boxed{\rho(g \cdot \mathbf{z}^{(n)}) g \cdot \mathbf{z}^{(n)} = \rho(z) \cdot \mathbf{z}^{(n)}; \text{ freeness} \Longrightarrow \rho(g \cdot \mathbf{z}^{(n)}) = \rho(\mathbf{z}^{(n)}) g^{-1}}$$

$$\Downarrow$$

ρ is a G-equivariant map.

c.-s. and ρ can be lifted to J^k for $k > n$:

$$\rho(\mathbf{z}^{(k)}) = \rho\left(\pi_n^k\left(\mathbf{z}^{(k)}\right)\right) \in G, \quad \mathcal{K}^k = \left(\pi_n^k\right)^{-1} \mathcal{K}^n \subset J^k, \quad \mathcal{K} = (\pi_n^\infty)^{-1} \mathcal{K}^n$$

Projection on the c.-s. $\gamma(\mathbf{z}) = \rho(\mathbf{z}) \cdot \mathbf{z} \colon J^\infty \longrightarrow \mathcal{K}$ for $\mathbf{z} \in J^\infty$.

Generalized definition of a moving frame.

$G \curvearrowright M$; a moving frame is an equivariant smooth map $\rho\colon M \to G$.

$$\begin{array}{ccc} G & \xrightarrow{R_{g^{-1}}} & G \\ \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\rho} \\ M & \xrightarrow[g]{} & M \end{array}$$

Thm. $\exists$ moving frame $\Leftrightarrow$ the action is regular and free.

Invariantization ι:

- **functions** $f \colon J^\infty \to \mathbb{R}$: $\boxed{\iota(f)(\mathbf{z}) = f(\rho(\mathbf{z}) \cdot \mathbf{z})}$

  $\{\mathcal{X}^i = \iota(x^i),\ I^\alpha_J = \iota(u^\alpha_J)\} \supset$ complete set of functionally indep. inv.

- **differential forms:** $\boxed{\iota(\Omega)|_{\mathbf{z}} = [\rho(\mathbf{z})]^* \left(\Omega|_{\rho(\mathbf{z})\cdot\mathbf{z}}\right)}$

  ω – 1-form, v – vector field: $\langle \iota(\omega)|_{\mathbf{z}}; v|_{\mathbf{z}} \rangle = \left\langle \omega|_{\rho(\mathbf{z})\cdot\mathbf{z}}; [\rho_{\mathbf{z}}]_* (v|_{\mathbf{z}}) \right\rangle$

  $\tilde{\theta}^\alpha_J = \iota(\theta^\alpha_J)$, $\alpha = 1, \ldots, q$ - inv. basis of contact 1-forms.
  $\tilde{\omega}^i = \iota(dx^i)$, $i = 1, \ldots, p$ - inv. basis of "horizontal" 1-forms.

- 1-st order inv. **diff. operators** = inv. vector fields.
  Basis – dual to $\{\tilde{\omega}^i, \tilde{\theta}^\alpha_J\}$.
  Basis of total inv. operators $\left\langle \tilde{\omega}^i; \tilde{\mathcal{D}}_j \right\rangle = \delta^i_j$.

Invariantization steps:

1. Pull back a form (a function) by the action of $g \in G$,
2. Replace g with $\rho(\mathbf{z})$.

Euclidean action on the plane curves:

$$x \mapsto \bar{x} = \cos(\phi)x - \sin(\phi)u + a, \quad u \mapsto \bar{u} = \sin(\phi)x + \cos(\phi)u + b$$

$$u_x \mapsto \bar{u}_x = \frac{\sin(\phi) + \cos(\phi)u_x}{\cos(\phi) - \sin(\phi)u_x}, \quad u_{xx} \mapsto \bar{u}_{xx} = \frac{u_{xx}}{(\cos(\phi) - \sin(\phi)u_x)^3}$$

loc. free and transitive on $J^1 \Rightarrow$ c.-s. $\mathcal{K}^1 = \{x = 0,\ u = 0,\ u_x = 0\}$

$$\Downarrow$$

moving frame is the solution of $\bar{x} = 0$, $\bar{u} = 0$, $\bar{u}_x = 0$:

$$\phi = -\arctan(u_x), \quad a = -\frac{x + u_x u}{\sqrt{u_x^2 + 1}}, \quad b = \frac{u_x x - u}{\sqrt{u_x^2 + 1}}$$

$$\Downarrow$$

$$g^* u_{xx} = \bar{u}_{xx} = \frac{u_{xx}}{(\cos(\phi) - \sin(\phi)u_x)^3} \quad \Rightarrow \quad I_2 = \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$g^*(dx) = \cos(\phi)dx - \sin(\phi)du \quad \Rightarrow \quad \tilde{\omega} = \frac{dx + u_x du}{\sqrt{1+u_x^2}} = \sqrt{1 + u_x^2}\, dx + \frac{u_x}{\sqrt{1+u_x^2}}\theta.$$

Structure equations.

$$d\,\iota\,(\Omega) = \iota\,(d\,\Omega) + \sum_{\kappa=1}^{r} \rho^*(\mu^\kappa) \wedge \iota\,[\mathbf{v}_\kappa\,(\Omega)]$$

$\mathbf{v}_1, \ldots \mathbf{v}_r$ – basis for infinitesimal generators of G-action

$\mathbf{v}_\kappa\,(\Omega)$ – Lie derivative of Ω with respect to $\mathbf{v}_\kappa$.

$\mu_1, \ldots \mu_r$– dual basis of invariant differential forms on G

Invariant bigrading $\Lambda^* = \bigoplus \tilde{\Lambda}^{s,t}$.
Unless the action is projectable $\tilde{\Lambda}^{s,t} \neq \Lambda^{s,t}$ and for $s \geq 1$:

$$d\colon \tilde{\Lambda}^{s,t} \longrightarrow \tilde{\Lambda}^{s+1,t} \oplus \tilde{\Lambda}^{s,t+1} \oplus \tilde{\Lambda}^{s-1,t+2} \Rightarrow d = d_{\tilde{H}} + d_{\tilde{V}} + d_W$$

$$d^2 = (d_{\tilde{H}} + d_{\tilde{V}} + d_W)^2 = 0$$

$$d_{\tilde{H}}^2 = 0, \quad d_{\tilde{V}}^2 + d_{\tilde{H}} d_W + d_W d_{\tilde{H}} = 0, \quad d_{\tilde{H}} \circ d_{\tilde{V}} = -d_{\tilde{V}} \circ d_{\tilde{H}}, \quad d_W^2 = 0$$

Algorithm for computing structure equations.

input: – dependent and independent variables $[x^1, \ldots, x^p], [u^1, \ldots, u^q]$,

– infinitesimal generators $V = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$,

– coordinate cross-section $\mathcal{K} = [z^{i_1} = c^1, \ldots, z^{i_r} = c^r]$,

– order of prolongation k.

output: – $d_{\tilde{V}} \tilde{\omega}^i, \quad d_{\tilde{H}} \tilde{\omega}^i = \sum c^i_{jk} \tilde{\omega}^j \wedge \tilde{\omega}^k, \quad [\tilde{\mathcal{D}}_k, \tilde{\mathcal{D}}_j] = -\sum c^i_{jk} \tilde{\mathcal{D}}_i$,

– $d_{\tilde{V}} \tilde{\theta}^\alpha_J, \quad d_{\tilde{H}} \tilde{\theta}^\alpha_J, \quad \tilde{\mathcal{D}}_i \left(\tilde{\theta}^\alpha_J \right)$,

– $d_{\tilde{V}} \mathcal{X}^i, \quad d_{\tilde{V}} I^\alpha_J, \quad d_{\tilde{H}} \mathcal{X}^i, \quad d_{\tilde{H}} I^\alpha_J, \quad \tilde{\mathcal{D}}_j \left(\mathcal{X}^i \right), \quad \tilde{\mathcal{D}}_j \left(I^\alpha_J \right)$.

operations: – differentiation and linear algebra.

comments: – explicit formulae for moving frame, invariants and invariant forms are not needed.

I. Preliminary computations:

- $V^k :=$ k-th prolongations of vectors V; $d := \dim J^k = p + qC^p_{p+k}$,
  $L := (r \times d)-$matrix of the coefficients of V^k,
  $L_\xi := (r \times p)$-submatrix of L, corresponding to x^i,
  $L_\phi := (r \times (d-p))$-submatrix of L, corresponding to u^α_J
  $L_c := r \times (d-p))$-matrix of contact forms such that
  $$\{L_c\}^\alpha_{J;\kappa} = \mathbf{v}_\kappa\,(\theta^\alpha_J) = d_V\,\phi^\alpha_{J;\kappa} - \sum_{i=1}^{p} u^\alpha_{Ji} d_V\,\xi^i_\kappa$$
  $L_r := (r \times r)$-submatrix of L of columns $i_1, \ldots, i_r$ defining $\mathcal{K}$.

- `if` $\det L_r|_{\mathcal{K}} = 0$ `then` *cross-section is invalid*; stop.
  `else` define the process of invariantization:
  $\iota(expression) := subs\Big(\{dx^i = \tilde{\omega}^i, \theta^\alpha_J = \tilde{\theta}^\alpha_J, z^{i_1} = c^1, \ldots, z^{i_r} = c^r,$
  for the rest of variables $x^i = \mathcal{X}^i, u^\alpha_J = I^\alpha_J\}, expression\Big)$

- matrices of variables and forms:

  $K := [z^{i_1}, \ldots, z^{i_r}](1 \times r)$- is matrix of normalized coordinates.

  $Z :=$ $(1 \times d)$- matrix of all coordinates,

  $I := \iota(Z)$ is $(1 \times d)$-matrix of invariants.

  $\tilde{\omega} := [\tilde{\omega}^1, \ldots, \tilde{\omega}^p]$ – invariant horizontal forms

  $\theta := [\theta^\alpha_J, \alpha = 1, \ldots, q, |J| \leq k]$ – $(1 \times d)$-matrix of contact forms.

  $\tilde{\theta} := [\tilde{\theta}^\alpha_J, \alpha = 1, \ldots, q, |J| \leq k] = \iota\,(\theta)$-invariant contact forms.

II. Structure equations:

- $$\begin{aligned} d_{\tilde{H}}\,\tilde{\omega} &= \iota\left[(d_H\,K) \wedge L_r^{-1}\,(d_H\,L_\xi)\right] \\ d_{\tilde{V}}\,\tilde{\omega} &= \iota\left[(d_H\,K) \wedge L_r^{-1}\,(d_V\,L_\xi) + (d_V\,K) \wedge L_r^{-1}\,(d_H L_\xi)\right] \end{aligned}$$

- $$\begin{aligned} d_{\tilde{H}}\,\tilde{\theta} &= \iota\left[d_H\,\theta - (d_H\,K) \wedge L_r^{-1} L_c\right] = \textstyle\sum_i \tilde{\mathcal{D}}_i\left(\tilde{\theta}\right) \wedge \tilde{\omega}^i \\ d_{\tilde{V}}\,\tilde{\theta} &= \iota\left[(d_V\,K) \wedge L_r^{-1} L_c\right] \end{aligned}$$

III. Differentials of fundamental invariants.

$$\begin{aligned} d_{\tilde{H}}\, I &= \iota\left[d_H\, Z - (d_H\, K)\, L_r^{-1} L\right] &= \sum_i \tilde{\mathcal{D}}_i\,(I) \wedge \tilde{\omega}^i \\ d_{\tilde{V}}\, I &= \iota\left[d_V\, Z - (d_V\, K)\, L_r^{-1} L\right] \end{aligned}$$

SE(2) acting on curves in R^2.

```
>   restart;with(Vessiot):read(IVBproc);with(linalg):
>   coord_frame([x],[u],fr1);
```

$$frame\ name:\ fr1$$

```
>   vectE:=[[1,0],[0,1],[-u[0],x]];
sectionE:=[x=0,u[0]=0,u[1]=0];
```

$$vectE := [[1, 0], [0, 1], [-u_0, x]]$$

$$sectionE := [x = 0, u_0 = 0, u_1 = 0]$$

```
>   ivb(vectE,sectionE,3);
```

$INVARIANT\ \ HORIZONTAL\ FORMS:$

$invariant\ \ vertical\ \ differentails$

$$\left[dv\,Dx\ \ = u_2\,Dx\ \ \wedge\ \ Cu_{[0]}\,\right]$$

$invariant\ \ horizontal\ \ differentails$

$$[0]$$

$$INVARIANT\ \ CONTACT\ \ FORMS$$

$$invariant\ \ vertical\ \ differentials:$$

$$\left[\begin{array}{c} dv\ Cu_{[0]} = u_2\ Cu_{[0]} \wedge\ Cu_{[1]}, \\ dv\ Cu_{[1]}\ = u_3\ Cu_{[0]}\ \wedge\ Cu_{[1]}, \\ dv\ Cu_{[2]} = u_4\ Cu_{[0]}\ \wedge\ Cu_{[1]}, \\ dv\ Cu_{[3]} = u_5\ Cu_{[0]}\ \wedge\ Cu_{[1]}\ - 6\, u_2\ Cu_{[1]}\ \wedge\ Cu_{[2]} \end{array}\right]$$

$$invariant\ \ horizontal\ \ differentials:$$

$$\left[\begin{array}{c} dh\ Cu_{[0]} = -{u_2}^2\ Dx\ \ \wedge\ Cu_{[0]}\ + Dx\ \wedge\ Cu_{[1]}\ , \\ dh\ Cu_{[1]}\ = -u_3\, u_2\ Dx\ \wedge\ Cu_{[0]}\ + Dx\ \wedge\ Cu_{[2]}\ , \\ dh\ Cu_{[2]}\ = -u_4\, u_2\ Dx\ \wedge\ Cu_{[0]}\ - 3\, {u_2}^2\ Dx\ \wedge\ Cu_{[1]}\ + Dx\ \wedge\ Cu_{[3]} \end{array}\right]$$

$$Lie\ \ derivatives\ \ w.r.t\ \ inv.\ \ diff.\ \ operators:$$

$$
\begin{bmatrix}
D_1\, Cu_{[0]} = -{u_2}^2\, Cu_{[0]} + Cu_{[1]} \\
D_1\, Cu_{[1]} = -u_3\, u_2\, Cu_{[0]} + Cu_{[2]} \\
D_1\, Cu_{[2]} = -u_4\, u_2\, Cu_{[0]} - 3\, {u_2}^2\, Cu_{[1]} + Cu_{[3]} \\
D_1\, Cu_{[3]} = -u_5\, u_2\, Cu_{[0]} - 4\, u_3\, u_2\, Cu_{[1]} - 6\, {u_2}^2\, Cu_{[2]} + Cu_{[4]}
\end{bmatrix}
$$

DIFFERENTIALS OF THE FUNDAMENTAL INVARIANTS

vertical

$$
\begin{bmatrix}
dv\, u_2 = Cu_{[2]} \\
dv\, u_3 = -3\, {u_2}^2\, Cu_{[1]} + Cu_{[3]}
\end{bmatrix}
$$

horizontal

$$\left[\begin{array}{c} dh\, u_2 = u_3\, Dx \\ \\ dh\, u_3 = \left(-3\, {u_2}^3 + u_4\right) Dx \end{array}\right]$$

$$Lie\ derivatives\ w.r.t\ inv.\ diff.\ operators:$$

$$\left[\begin{array}{c} \mathrm{D}_1\, u_2 = u_3 \\ \\ \mathrm{D}_1\, u_3 = -3\, {u_2}^3 + u_4 \end{array}\right]$$

SE(2) acting on independent variables for surfaces in R^3.

```
>  coord_frame([x,y],[u],fr1):
>  vectors:=[[1,0,0],[0,1,0],[-y,x,0]];
section:=[x=0,y=0,u[1,0]=0];
>  ivb(vectors,section,2);
```

INVARIANT HORIZONTAL FORMS :

invariant vertical differentails

$$\left[dv\,Dx = \frac{Dy \wedge Cu_{[1,0]}}{u_{0,1}},\ dv\,Dy = -\frac{Dx \wedge Cu_{[1,0]}}{u_{0,1}}\right]$$

invariant horizontal differentails

$$\left[dh\,Dx = -\frac{u_{2,0}\,Dx \wedge Dy}{u_{0,1}},\ dh\,Dy = -\frac{u_{1,1}\,Dx \wedge Dy}{u_{0,1}}\right]$$

$$Lie\ algebra\ of\ invariant\ differential\ operators:$$

```
> invD := [[Lie_alg, InvH, [p]],
[seq(invDstr[t],t=1..p*(p^2-p)/2)]];
>
Lie_alg_init(invD,[Delta],[omega]):frameInformation():
```

$$invD := \left[[Lie\_alg, InvH, [2]], \left[\left[[1, 2, 1], \frac{u_{2,0}}{u_{0,1}}\right], \left[[1, 2, 2], \frac{u_{1,1}}{u_{0,1}}\right]\right]\right]$$

$$frame\ name: \quad InvH$$

$$library\ name: \quad Koszul\_frame$$

$$Frame\ Jet\ Variables:$$

$$[z1, z2]$$

$$Frame\ Labels$$

$$[\Delta 1\ , \Delta 2\ ]$$

$$CoFrame\ \ Labels$$

$$[\omega 1\ \ ,\ \omega 2\ \ ]$$

$$Horizontal\ \ Coframe\ \ Labels$$
$$[]$$
$$Vertical\ \ Coframe\ \ Labels$$
$$[]$$

$$ext\_d\ :,\ \omega 1,\ -\frac{u_{2,\,0}\,\omega 1\ \ \wedge\ \ \omega 2}{u_{0,\,1}}$$

$$ext\_d\ :,\ \omega 2,\ -\frac{u_{1,\,1}\,\omega 1\ \ \wedge\ \ \omega 2}{u_{0,\,1}}$$

Lie derivatives of invariants w.r.t inv. diff. operators :

$$\begin{bmatrix} \mathrm{D}_1\, u_{0,\,0} = 0 & \mathrm{D}_2\, u_{0,\,0} = u_{0,\,1} \\ \mathrm{D}_1\, u_{0,\,1} = u_{1,\,1} & \mathrm{D}_2\, u_{0,\,1} = u_{0,\,2} \\ \mathrm{D}_1\, u_{2,\,0} = -2\,\frac{u_{1,\,1}\, u_{2,\,0}}{u_{0,\,1}} + u_{3,\,0} & \mathrm{D}_2\, u_{2,\,0} = -2\,\frac{{u_{1,\,1}}^2}{u_{0,\,1}} + u_{2,\,1} \\ \mathrm{D}_1\, u_{1,\,1} = \frac{{u_{2,\,0}}^2 - u_{2,\,0}\, u_{0,\,2}}{u_{0,\,1}} + u_{2,\,1} & \mathrm{D}_2\, u_{1,\,1} = \frac{u_{1,\,1}\,(u_{2,\,0} - u_{0,\,2})}{u_{0,\,1}} + u_{1,\,2} \\ \mathrm{D}_1\, u_{0,\,2} = 2\,\frac{u_{1,\,1}\, u_{2,\,0}}{u_{0,\,1}} + u_{1,\,2} & \mathrm{D}_2\, u_{0,\,2} = 2\,\frac{{u_{1,\,1}}^2}{u_{0,\,1}} + u_{0,\,3} \end{bmatrix}$$

$$\textit{syzygies :}\quad D_2 u_{2,\,0} - D_1 u_{1,\,1} = \frac{2\,{u_{1,\,1}}^2 + {u_{2,\,0}}^2 - u_{2,\,0}\, u_{0,\,2}}{u_{0,\,1}}$$

$$D_2 u_{1,\,1} - D_1 u_{0,\,2} = -\frac{u_{1,\,1}\,(u_{2,\,0} + u_{0,\,2})}{u_{0,\,1}}$$

Invariant integration by parts for plane curves.

$G \curvearrowright J(\mathbb{R}^2, 1)$, $\tilde{\omega}$ – invariant differential form, $\tilde{\mathcal{D}}$ – dual invariant differential operator, κ – generating invariant, $\kappa_i = \left(\tilde{\mathcal{D}}\right)^i \kappa$.

$$\tilde{\lambda} = \tilde{L}\left(\kappa, \kappa_1, \ldots, \kappa_n\right)\tilde{\omega}$$

$$d_{\tilde{V}}\tilde{\lambda} = d_{\tilde{V}}\tilde{L} \wedge \tilde{\omega} + \tilde{L}\, d_{\tilde{V}}\tilde{\omega} = \sum_{i=0}^{n} \frac{\partial \tilde{L}}{\partial \kappa_i}\left(d_{\tilde{V}}\kappa_i\right) \wedge \tilde{\omega} + \tilde{L}\, d_{\tilde{V}}\tilde{\omega} \equiv (\mathrm{mod}\ d_{\tilde{H}})$$

$$\boxed{\left(d_{\tilde{V}}\kappa_i\right) \wedge \tilde{\omega} = \left(d_{\tilde{V}}\tilde{\mathcal{D}}\kappa_{i-1}\right) \wedge \tilde{\omega} = \left(d_{\tilde{V}} d_{\tilde{H}}\kappa_{i-1}\right) - \kappa_i d_{\tilde{V}}\tilde{\omega} = -\left(d_{\tilde{H}} d_{\tilde{V}}\kappa_{i-1}\right) - \kappa_i d_{\tilde{V}}\tilde{\omega}}$$

$$\begin{aligned} &\equiv \quad \textstyle\sum_{i=0}^{n} d_{\tilde{H}}\left(\frac{\partial \tilde{L}}{\partial \kappa_i}\right)\left(d_{\tilde{V}}\kappa_{i-1}\right) - \left(\sum_{i=0}^{n} \frac{\partial \tilde{L}}{\partial \kappa_i}\kappa_i - \tilde{L}\right) d_{\tilde{V}}\tilde{\omega} \\ &= \quad \textstyle\sum_{i=0}^{n}\left(-\tilde{\mathcal{D}}\right)\left(\frac{\partial \tilde{L}}{\partial \kappa_i}\right)\left(d_{\tilde{V}}\kappa_{i-1}\right) \wedge \tilde{\omega} - \left(\ldots\right) d_{\tilde{V}}\tilde{\omega}. \end{aligned}$$

repeat!

$$\boxed{d_{\tilde{V}}\tilde{\lambda} \equiv \mathcal{E}\left(\tilde{L}\right) d_{\tilde{V}}\kappa \wedge \tilde{\omega} - \mathcal{H}\left(\tilde{L}\right) d_{\tilde{V}}\tilde{\omega}.}$$

$$\mathcal{E}\left(\tilde{L}\right) = \sum_{i=0}^{n} \left(-\tilde{\mathcal{D}}\right)^{i} \frac{\partial \tilde{L}}{\partial \kappa_i}, \quad \mathcal{H}\left(\tilde{L}\right) = \sum_{i>j\geq 0}^{n} \kappa_{i-j} \left(-\tilde{\mathcal{D}}\right)^{j} \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

$$\lambda = L(x, u, u_1, \ldots, u_m)\, dx \quad \leftrightarrow \quad \tilde{\lambda} = \tilde{L}\left(\kappa, \kappa_1, \ldots, \kappa_n\right) \tilde{\omega}$$

$$d_V(dx) = 0 \qquad\qquad d_{\tilde{V}}(\tilde{\omega}) = \mathcal{B}(\tilde{\theta}_0) \wedge \tilde{\omega}$$

$$d_V(u) = \theta_0 \qquad\qquad d_{\tilde{V}}(\kappa) = \mathcal{A}(\tilde{\theta}_0)$$

$$\boxed{d_{\tilde{V}}\tilde{\lambda} \equiv \left[\mathcal{A}^{*}\mathcal{E}\left(\tilde{L}\right) - B^{*}\mathcal{H}\left(\tilde{L}\right)\right] \tilde{\theta}_0 \wedge \tilde{\omega}.}$$

Euclidean group $SE(2) = SO(2) \ltimes R^2$ acts on plane curves $u = u(x)$:

$$\boxed{x \mapsto \cos(\phi)x - \sin(\phi)u + a, \quad u \mapsto \sin(\phi)x + \cos(\phi)u + b.}$$

- $d_{\tilde{V}}\kappa = \tilde{\theta}_2 = \frac{d}{ds}\theta_1 + \kappa^2\tilde{\theta}_0 = \left[\left(\frac{d}{ds}\right)^2 + \kappa^2\right]\tilde{\theta}_0,$
- $d_{\tilde{V}}\tilde{\omega} = -\kappa\,\tilde{\theta}_0 \wedge \tilde{\omega}.$

$$\boxed{d_{\tilde{V}}\lambda \equiv= \left\{\left[\left(\frac{d}{ds}\right)^2 + \kappa^2\right]\mathcal{E}\left(\tilde{L}\right) + \kappa\mathcal{H}\left(\tilde{L}\right)\right\}\tilde{\theta}_0 \wedge \tilde{\omega}}$$