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Invariants: Computation and Applications

Tutorial

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https://iakogan.math.ncsu.edu/suppl/issac23.html for Maple worksheets with codes and examples.

Overview

Foundation:

- $\boxed{\text{Basic definitions and examples}}$
- Motivation and applications
- Smooth vs. algebraic setup
- Binary forms under linear changes of variables
- Planar curves under rigid motions
- History remarks

Cross-section method for computing invariants:

- Smooth construction
- Algebraic construction

Differential signature construction for solving the equivalence problem:

- Planar curves
- Binary forms

What is invariant theory?

It is a study of orbits and invariants defined by a group action.

Groups and their actions:

- A group $\mathcal{G}$ is a set with an identity element $e \in \mathcal{G}$ and two maps:
  - group operation: $m \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, (notation: $m(\mathrm{g}_1, \mathrm{g}_2) = \mathrm{g}_1 \cdot \mathrm{g}_2$), such that

    i. $\quad e \cdot \mathbf{g} = \mathbf{g} \cdot e, \quad \forall \mathbf{g} \in \mathcal{G};$

    ii. $\quad (\mathrm{g}_1 \cdot \mathrm{g}_2) \cdot \mathrm{g}_3 = \mathrm{g}_1 \cdot (\mathrm{g}_2 \cdot \mathrm{g}_3), \quad \forall \mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3 \in \mathcal{G}.$

    Proposition: e is unique.
  - inversion: $i \colon \mathcal{G} \to \mathcal{G}$, (notation: $i(\mathrm{g}) = \mathrm{g}^{-1}$, the inverse of g), such that

$$\mathbf{g}^{-1} \cdot \mathbf{g} = \mathbf{g} \cdot \mathbf{g}^{-1} = e.$$

    Proposition: $i^2 = \mathrm{Id}|_{\mathcal{G}}$. (In other words $(\mathrm{g}^{-1})^{-1} = \mathrm{g}$.)

- An action of a group $\mathcal{G}$ on a set $\mathcal{Z}$ is a map $\alpha\colon \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$

  (notation: $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}, \qquad \alpha(\mathrm{g}, \mathbf{z}) = \mathrm{g}\,\mathbf{z}$), such that

  i. $\alpha(e, \mathbf{z}) = \mathbf{z},\ \forall \mathbf{z} \in \mathcal{Z};$

  ii. $\alpha(\mathrm{g}_1 \cdot \mathrm{g}_2, \mathbf{z}) = \alpha(\mathrm{g}_1, \alpha(\mathrm{g}_2, \mathbf{z})),\ \forall \mathbf{z} \in \mathcal{Z}, \mathrm{g}_1, \mathrm{g}_2 \in \mathcal{G}.$

- The orbit of $\mathbf{z} \in \mathcal{Z}$ under an action α is a set

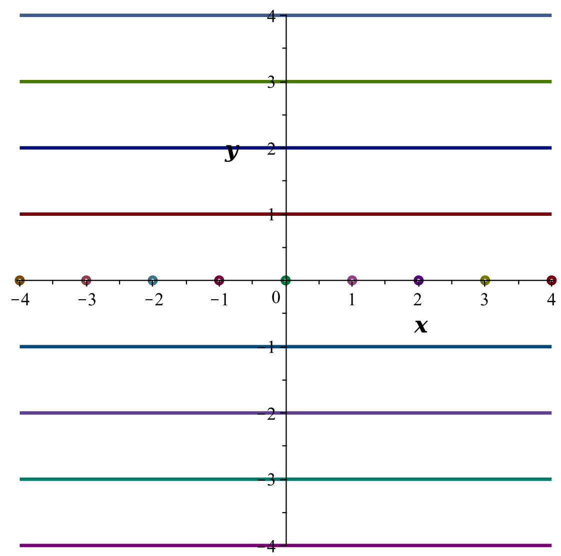
$$\mathcal{O}_{\mathbf{z}} = \{\tilde{\mathbf{z}} \in \mathcal{Z} | \exists \mathrm{g} \in \mathcal{G} : \tilde{\mathbf{z}} = \alpha(\mathrm{g}, \mathbf{z})\}.$$

- $\mathcal{O}_{\mathbf{z}}$ is the image of $\mathcal{G}$ under the map $\alpha_{\mathbf{z}}\colon \mathcal{G} \to \mathcal{Z}, \quad \alpha_{\mathbf{z}}(\mathrm{g}) = \alpha(\mathrm{g}, \mathbf{z})$.
- If $\forall \mathbf{z} \in \mathcal{Z}$, $\alpha_{\mathbf{z}}$ is injective, the action is called free.

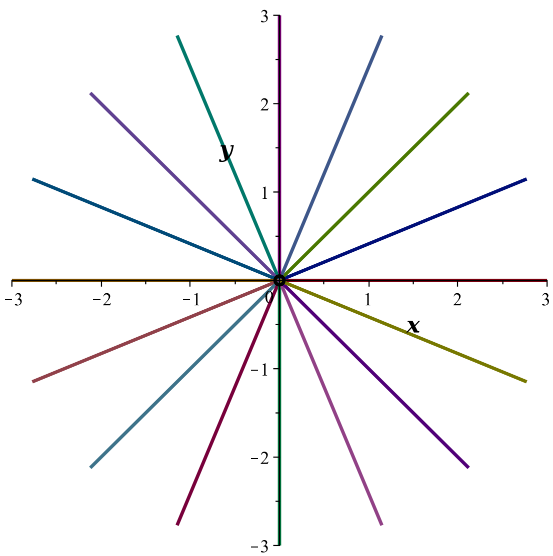
Example: four actions of $\mathcal{G} = \mathbb{R}$ (with $\cdot = +$) on $\mathcal{Z} = \mathbb{R}^2$

For $t \in \mathcal{G}$ and $(x, y) \in \mathcal{Z}$:

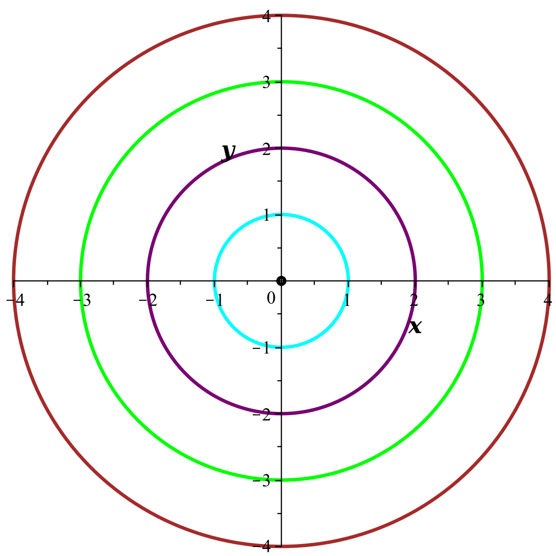
$$\alpha_1(t, (x, y)) = (x + ty,\ y) \qquad \text{(shear)}$$
$$\alpha_2(t, (x, y)) = \left(e^t x,\ e^t y\right), \qquad \text{(positive scaling)}$$
$$\alpha_3(t, (x, y)) = (x \cos t - y \sin t,\ x \sin t + y \cos t) \qquad \text{(rotation)}$$
$$\alpha_4(t, (x, y)) = \left(e^t\ (x \cos t - y \sin t),\ e^t\ (x \sin t + y \cos t)\right)$$



orbits of α_1



orbits of α_2



orbits of α_3

Describe the orbits of α_4. Is any of these actions free?

Motivation to study orbits:

- Orbits under an action $\alpha \colon \mathcal{G} \curvearrowright \mathcal{Z}$ partition $\mathcal{Z}$ into equivalence classes:

$$\mathbf{z}_1 \underset{\alpha}{\cong} \mathbf{z}_2 \Longleftrightarrow \mathcal{O}_{\mathbf{z}_1} = \mathcal{O}_{\mathbf{z}_2}.$$

- Many classification problems in math and beyond can be restated as problems of orbit classification, or the study of the orbit space $\mathcal{Z}/\mathcal{G}$.

How to describe $\mathcal{Z}/\mathcal{G}$?

Invariants and orbit separation

A function (or a map) $f\colon \mathcal{Z} \to \mathcal{Y}$ is invariant under an action $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}$ if it is constant along each orbit:

$$f(\alpha(\mathbf{g}, \mathbf{z})) = f(\mathbf{z}) \text{ for all } \mathbf{z} \in \mathcal{Z}, \mathbf{g} \in \mathcal{G}.$$

“Invariant” is often used as a noun rather than as an adjective e.g:

“polynomial invariant” =“invariant polynomial function”.

A set $\mathcal{F}$ of invariant functions under an action $\alpha\colon \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$ separates orbits on $\mathcal{U} \subset \mathcal{Z}$ if for any $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{U}$:

$$\forall f \in \mathcal{F}, \quad f(\mathbf{z}_1) = f(\mathbf{z}_2) \Longleftrightarrow \mathcal{O}_{\mathbf{z}_1} = \mathcal{O}_{\mathbf{z}_2}.$$

Any constant function is invariant, but constant functions can’t separate orbits.

Invariance

$\alpha : \mathcal{G} \curvearrowright \mathcal{Z}$

$$\begin{tikzcd}
\mathcal{Z} \arrow[rr, "\alpha_{\mathbf{g}}"] \arrow[dr, "f"'] & & \mathcal{Z} \arrow[dl, "f"] \\
& \mathcal{Y} &
\end{tikzcd}$$

Equivariance

$\alpha : \mathcal{G} \curvearrowright \mathcal{Z}$ and $\beta : \mathcal{G} \curvearrowright \mathcal{Y}$

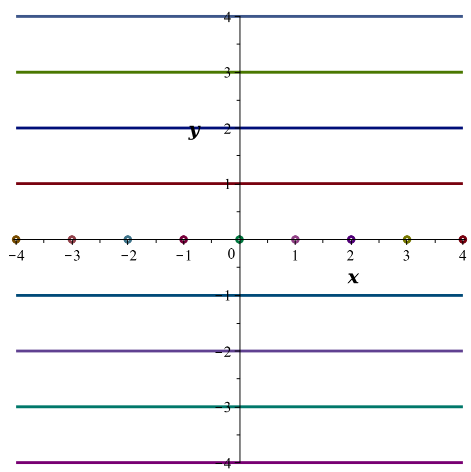
$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\alpha_{\mathbf{g}}"] \arrow[d, "f"'] & \mathcal{Z} \arrow[d, "f"] \\
\mathcal{Y} \arrow[r, "\beta_{\mathbf{g}}"] & \mathcal{Y}
\end{tikzcd}$$

$\alpha_{\mathbf{g}}(\mathbf{z}) = \alpha(\mathbf{g}, \mathbf{z})$ and $\beta_{\mathbf{g}}(\mathbf{z}) = \beta(\mathbf{g}, \mathbf{z})$

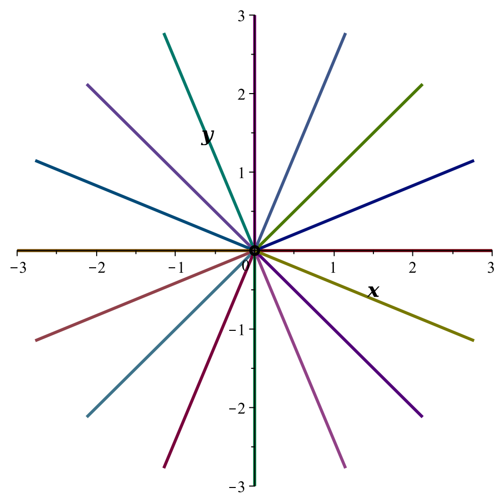
- Find invariant function(s) $\mathbb{R}^2 \to \mathbb{R}$ for the three actions of $\mathcal{G} = \mathbb{R}$ (with $\cdot = +$) on $\mathcal{Z} = \mathbb{R}^2$.

- On what subset of $\mathbb{R}^2$ do these invariants separate orbits?

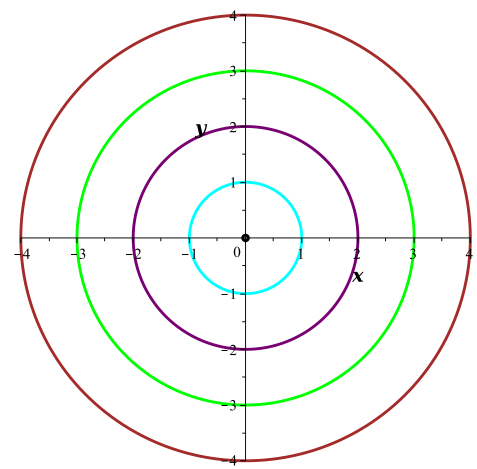
$\alpha_1\,(t,(x,y)) = (x + ty,\ y)$



$\alpha_2(t,(x,y)) = \left(e^t\,x,\ e^t\,y\right)$

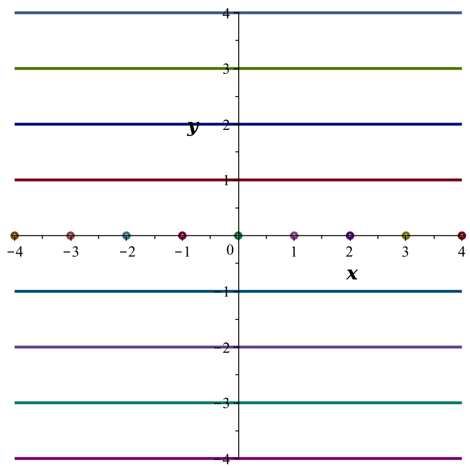


α_3 - rotations

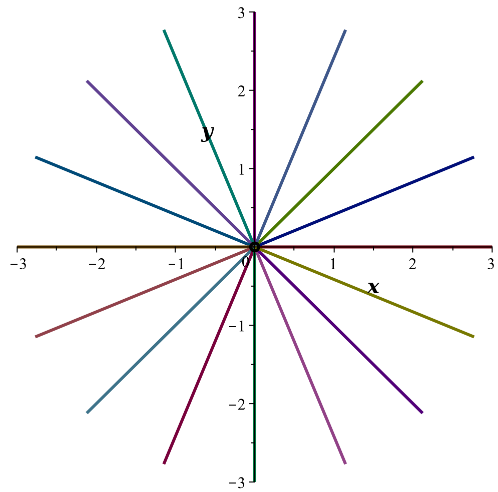


Invariants for actions of $\mathcal{G} = \mathbb{R}$ (with $\cdot = +$) on $\mathcal{Z} = \mathbb{R}^2$

$\alpha_1\,(t,(x,y)) = (x + ty,\, y)$



$\alpha_2(t,(x,y)) = \left(e^t\, x,\, e^t\, y\right)$



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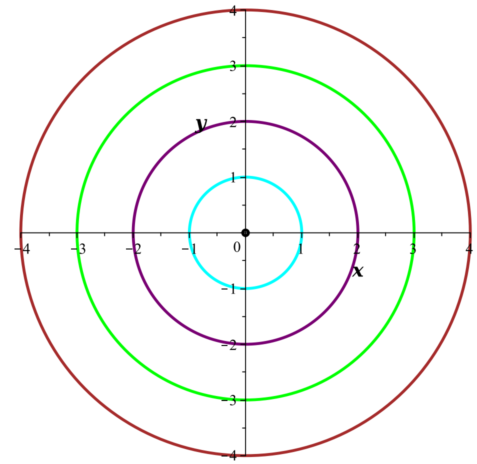


- $v(x,y) = y$ separates orbits away from the x-axis.

- No non-constant continuous invariants.
- $h(x,y) = \frac{y}{x}$ separates orbits in the right (or left) half plane.
- $\mathcal{F} = \left\{ \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\}$ separates orbits away from the origin.

- $R(x,y) = x^2 + y^2$ separates all orbits.

Some terminology:

- The stabilizer of $\mathbf{z} \in \mathcal{Z}$ under an action α is the set

$$\mathcal{G}_{\mathbf{z}} = \{\mathbf{g} \in \mathcal{G} \,|\, \alpha(\mathbf{g}, \mathbf{z}) = \mathbf{z}\}.$$

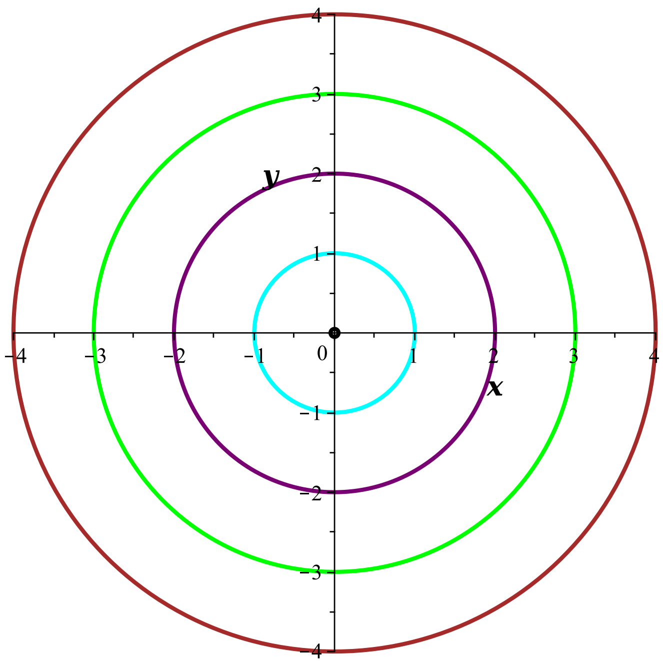
  - Proposition: $\mathcal{G}_{\mathbf{z}}$ is a subgroup of $\mathcal{G}$.
  - Observation: for a free action, the stabilizer of every point is $\{e\}$.

- The intersection of all stabilizers $\mathcal{G}_{\mathcal{Z}} = \cap_{\mathbf{z} \in \mathcal{Z}} \mathcal{G}_{\mathbf{z}}$ is the global stabilizer.

  If $\mathcal{G}_{\mathcal{Z}} = \{e\}$, the action is called effective.

  Proposition: $\mathcal{G}_{\mathcal{Z}}$ is a <u>normal</u> subgroup of $\mathcal{G}$. Induced action of $\mathcal{G}/\mathcal{G}_{\mathcal{Z}}$ is effective.

Example: $\alpha_3\,(t,(x,y)) = (x\cos t - y\sin t,\; x\sin t + y\cos t)$



Find:

- the stabilizer $\mathcal{G}_z$, for each $\mathbf{z} \in \mathbb{R}^2$,
- the global stabilizer $\mathcal{G}_{\mathcal{Z}}$.
- the quotient $\mathcal{G}\backslash\mathcal{G}_{\mathcal{Z}}$.

- For $\mathbf{z} = (0,0)$, $\mathcal{G}_{\mathbf{z}} = \mathbb{R}$, for $\mathbf{z} \neq (0,0)$, $\mathcal{G}_{\mathbf{z}} = \{2\pi n \,|\, n \in \mathbb{Z}\}$

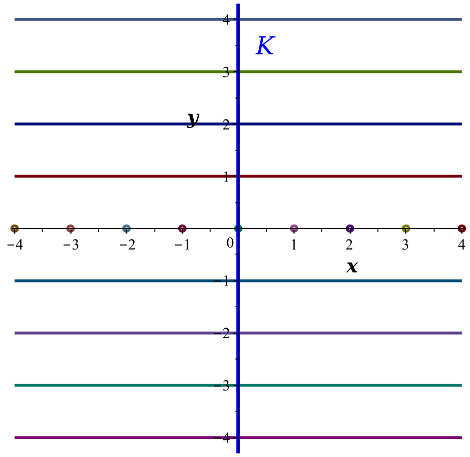
- $\mathcal{G}_{\mathcal{Z}} = \{2\pi n \,|\, n \in \mathbb{Z}\}$.

- $\mathcal{G}/\mathcal{G}_{\mathcal{Z}} \cong SO_2(\mathbb{R}) = \left\{ \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \,\middle|\, c^2 + s^2 = 1 \right\}$ is the special orthogonal group.

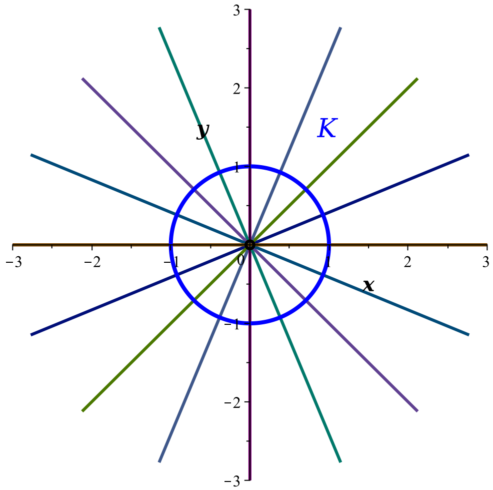
Canonical forms – a choice of a “simple” representative of each orbit.

“Simple” is in the eyes of the beholder.

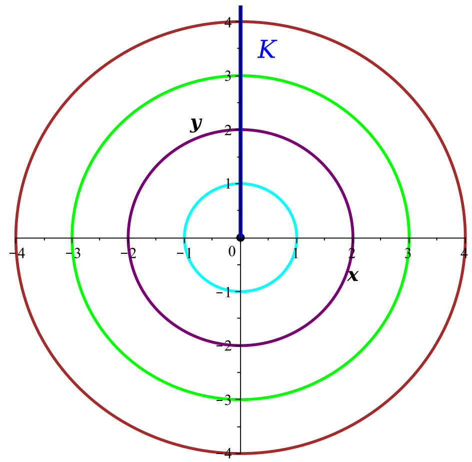
$\alpha_1\,(t,(x,y)) = (x + ty,\, y)$



$\alpha_2(t,(x,y)) = \left(e^t\, x,\, e^t\, y\right)$



α_3 - rotations



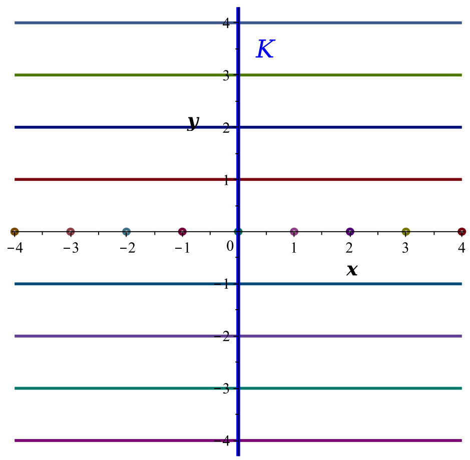
Let $\tilde{\mathbf{z}} = (\iota x, \iota y)$ be the canonical form $\mathbf{z} = (x, y)$.

Express $(\iota x, \iota y)$ in terms of (x, y).

- For α_1, $\quad (\iota x, \iota y) = (0, y)$, when $y \neq 0$.
- For α_2, $\quad (\iota x, \iota y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$, when $(x, y) \neq (0, 0)$.
- For α_3, $\quad (\iota x, \iota y) = (0, \sqrt{x^2 + y^2})$.

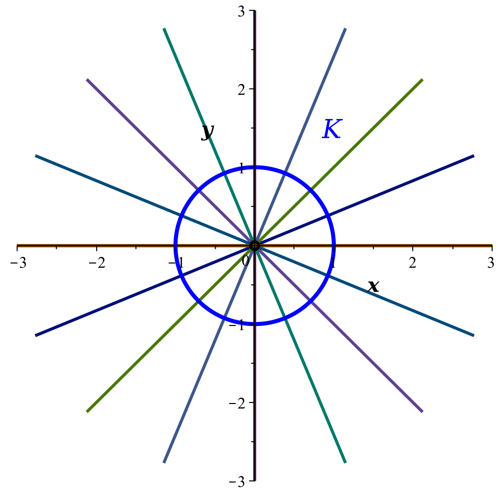
Normalized invariants – coordinates of the canonical form.

$\alpha_1\,(t,(x,y)) = (x+ty,\, y)$



$(\iota x, \iota y) = (0, y)$, when $y \neq 0$.

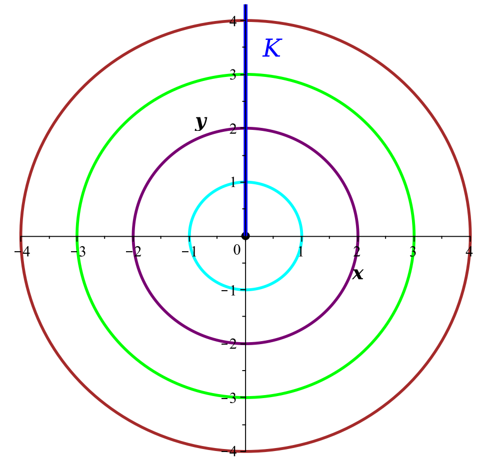
$\alpha_2(t,(x,y)) = \left(e^t\, x,\, e^t\, y\right)$



$(\iota x, \iota y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right),$

when $(x,y) \neq (0,0)$

α_3 - rotations



$(\iota x, \iota y) = (0, \sqrt{x^2+y^2})$.

Replacement property of normalized invariants $(\iota x, \iota y)$:

- Any function $f(x,y)$ can be turned into an invariant function

$$\iota f(x,y) = f(\iota x, \iota y).$$

- If f is invariant then $\iota f = f$.

Example: Jordan canonical forms (Camille Jordan, (1870)).

Matrices under adjoint (similarity) transformations:

- $\mathcal{Z} = M_n(\mathbb{C})$ is a set of $n \times n$ matrices,
- $\mathcal{G} = GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices.

$$\alpha(\mathbf{g}, \mathbf{z}) = \mathbf{g}\mathbf{z}\mathbf{g}^{-1}$$

$$\begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & & & & \\ & & \lambda & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}, \ldots$$

For a generic matrix (with n linear independent eigenvectors) the canonical form is diagonal and normalized invariants are

$$\iota \mathbf{z}_{ij} = \begin{cases} \lambda_i, & i = j \quad (eigenvalues) \\ 0, & i \neq j \end{cases}$$

- Eigenvalues are the roots of the characteristic polynomial:

$$\det(\lambda I - \mathbf{z}) = \lambda^n + \sigma_1 \lambda^{n-1} + \cdots + \sigma_{n-1}\lambda + \sigma_n$$

- The coefficients of the characteristic polynomial are invariant polynomials in entries z_{ij} of $\mathbf{z}$:

$$\{\sigma_1 = \mathrm{tr}(\mathbf{z}), \ldots, \sigma_{\mathrm{n}} = \det(\mathbf{z})\}$$

- Using replacement properties of normalized invariants we can recover the standard expressions of σ_i in terms of the eigenvalues:

$$\{\sigma_1 = \lambda_1 + \cdots + \lambda_n, \ldots, \sigma_n = \lambda_1 \cdots \lambda_n\}$$

The the cross-section method in the $\boxed{\text{second part}}$ of the tutorial provides a proper framework and algorithms for finding normalized invariants, as well as generators the field of rational invariants.

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Cross-section method for computing invariants:

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- Algebraic construction

Differential signature construction for solving the equivalence problem:

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Motivation to study orbits:

- Orbits under an action $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}$ partition $\mathcal{Z}$ into equivalence classes:

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- Many classification problems in math and beyond can be restated as problems of orbit classification, or the study of orbit space $\mathcal{Z}/\mathcal{G}$.

Motivation to study invariants:

- Canonical forms are expressed in terms of invariants $\rightarrow$ invariants are the key for describing the orbit space.

- If a problem has a symmetry (math, physics, etc), rewriting it in terms of invariants (symmetry reduction) showcases the essence of the problem and often simplifies it.

How to compute invariants needed to describe canonical forms?

How to rewrite and manipulate a problem in terms of invariants?

A snapshot of applications:

Computer vision and image processing

- E. Calabi, P. J. Olver, C. Shakiban, A. Tannenbaum, and S. Haker. ”Differential and numerically invariant signature curves applied to object recognition.” Int. J. Computer Vision (1998).
- S. Feng, H. Krim, and I. A. Kogan. ”3D face recognition using Euclidean integral invariants signature”. IEEE/SP workshop (2007).
- D.G. Lowe. ”Distinctive Image Features from Scale-Invariant Keypoints.” Int. J. Computer Vision” (2004).

Mechanics

- M. Olive, B. Kolev, R. Desmorat, and B. Desmorat, ”Characterization of the symmetry class of an elasticity tensor using polynomial covariants”. Math. Mech. Solids (2022)

Mathematical physics

- N.H. Ibragimov, ”Transformation groups applied to mathematical physics.” Springer. 1985.
- P.J. Olver ”Applications of Lie groups to differential equations”. Springer. 1986.

Variational calculus

- I. A. Kogan and P.J. Olver, ”Invariant Euler-Lagrange equations and the invariant variational bicomplex.” Acta Applicandae Math. (2003).

Dynamical system

- E. Hubert and G. Labahn. ”Scaling invariants and symmetry reduction of dynamical systems”. Foundations of Computational Mathematics, (2013).

Numerical Analysis

- E. L. Mansfield, A. Rojo-Echeburúa, P. E. Hydon, and L. Peng. ”Moving frames and Noether’s finite difference conservation laws” Trans. Math. Appl. (2019).

- A. Bihlo and F. Valiquette. ”Symmetry-preserving finite elements schemes: An introductory investigation”, SIAM J. Sci. Comput. (2019)

Math Biology

- T. Flash and A. A. Handzel. ”Affine differential geometry analysis of human arm movements”. Biological cybernetics, (2007).

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Our smooth setup: (can be generalized)

- $\mathcal{G}$ is a real Lie group of positive dimension.
- $\mathcal{Z} \underset{open}{\subset} \mathbb{R}^n$ with metric topology.
- $\alpha\colon \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$ is a C^∞-smooth map.
- The domain of an action may be restricted to $\Omega \underset{open}{\subset} \mathcal{G} \times \mathcal{Z}$ (local action).
- A point $\mathbf{z}$ is called generic if its orbit has the maximal dimension.
- Invariants are C^∞-smooth functions $\mathcal{Z} \to \mathbb{R}$.
- Notation: $\boxed{C^\infty(\mathcal{Z})^{\mathcal{G}} \subset C^\infty(\mathcal{Z})}$ (smooth invariants as a subset of smooth functions.)
- The domain of invariants may be restricted to $\mathcal{U} \underset{open}{\subset} \mathcal{Z}$ and they may be invariant only for $\mathbf{g}$ in some open neighborhood of $e \in \mathcal{G}$ (local invariants).

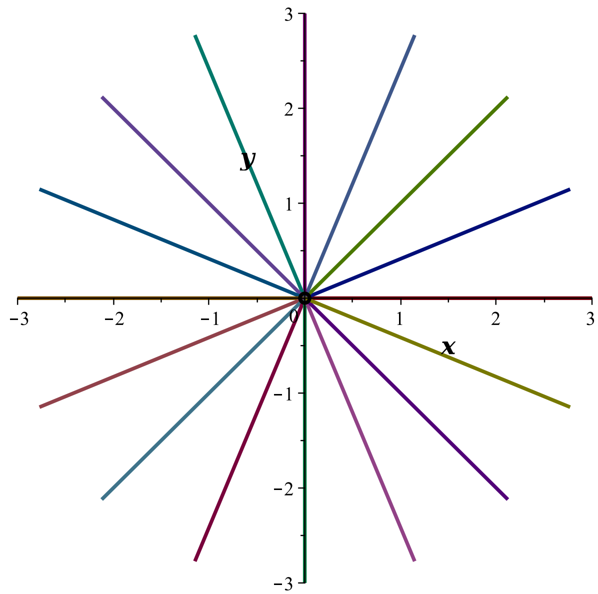
Smooth structural result:

Theorem (Corollary of Frobenius Theorem, Feodor Deahna (1840))

For a generic point $\mathbf{z}$, there exists an open neighborhood $\mathcal{U} \subset \mathcal{Z}$, such that $C^\infty(\mathcal{U})^\mathcal{G}$ is functionally generated by $n - s$ functionally independent local invariants (called fundamental invariants) where $n = \dim \mathcal{Z}$ and $s = \dim \mathcal{O}_\mathbf{z}$.

Example

$$\alpha_2(t,(x,y)) = \left(e^t\, x,\; e^t\, y\right)$$



For each of the three invariants

$h(x,y) = \frac{y}{x}$,

$f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$,

$g(x,y) = \frac{y}{\sqrt{x^2+y^2}}$,

find a set $\mathcal{U}$, where this invariant is fundamental.

Our algebraic setup: (can be generalized)

- $\mathcal{G}$ is an affine algebraic group of positive dimension over a field $\mathbb{K}$ of characteristic zero.

- $\mathcal{Z} = \mathbb{K}^n$ with Zariski topology.

- A statement is true for a generic $\mathbf{z}$ if it is true on a Zariski open subset.

- $\alpha \colon \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$ is a rational map.

  More precisely, we consider the restriction of a rational action over $\overline{\mathbb{K}}$.

- Invariants are functions $\mathcal{Z} \to \mathbb{K}$
  - $\boxed{\mathbb{K}[\mathcal{Z}]^{\mathcal{G}} \subset \mathbb{K}[\mathcal{Z}]}$
    (the ring of polynomial invariants as a subring of all polynomial functions.)
  - $\boxed{\mathbb{K}(\mathcal{Z})^{\mathcal{G}} \subset \mathbb{K}(\mathcal{Z})}$
    (the field of rational invariants as a subfield of all rational functions.)
  - $\boxed{\overline{\mathbb{K}(\mathcal{Z})}^{\mathcal{G}} \subset \overline{\mathbb{K}(\mathcal{Z})}}$
    (the field of algebraic invariants as a subfield of algebraic functions over $\mathbb{K}(\mathcal{Z})$.)

Algebraic structural results:

Finiteness Theorems:

- If $\mathcal{G}$ is reductive, the ring $\mathbb{K}[\mathcal{Z}]^{\mathcal{G}}$ is finitely generated (Hilbert, 1890).
- The field $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ is finitely generated (Rosenlicht, 1956).

Definition: A set $\mathcal{F}$ of functions on $\mathcal{Z}$ is generically separating, if there exists a Zariski open subset of $\mathcal{Z}$, on which $\mathcal{F}$ separates orbits.

Theorem: Over an algebraically closed field $\overline{\mathbb{K}}$, a subset $\mathcal{F} \subset \overline{\mathbb{K}}(\mathcal{Z})^{\mathcal{G}}$ of rational invariants is

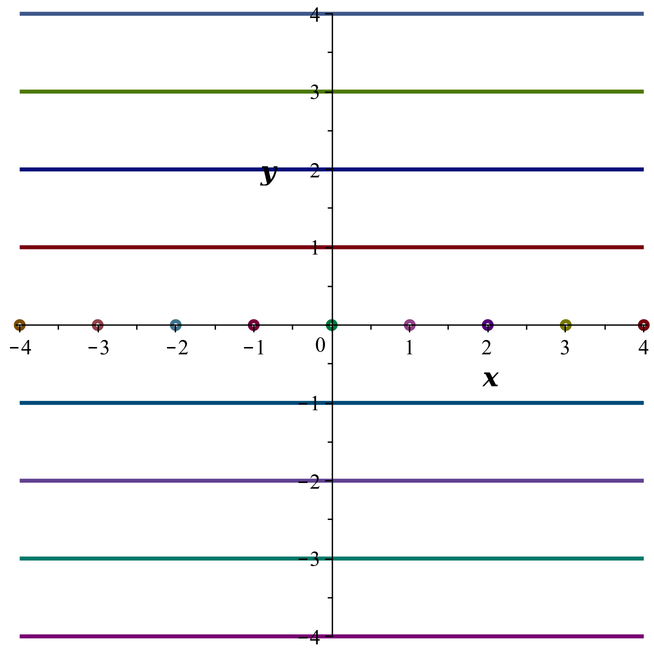
$$\text{generating} \quad \Longleftrightarrow \quad \text{generically separating.}$$

M. Rosenlicht (1956), V. L. Popov and E. B. Vinberg. (1994).

Both directions are $\boxed{\text{false}}$ over a non-algebraically closed field!

$\mathbb{R} \curvearrowright \mathbb{R}^2$:

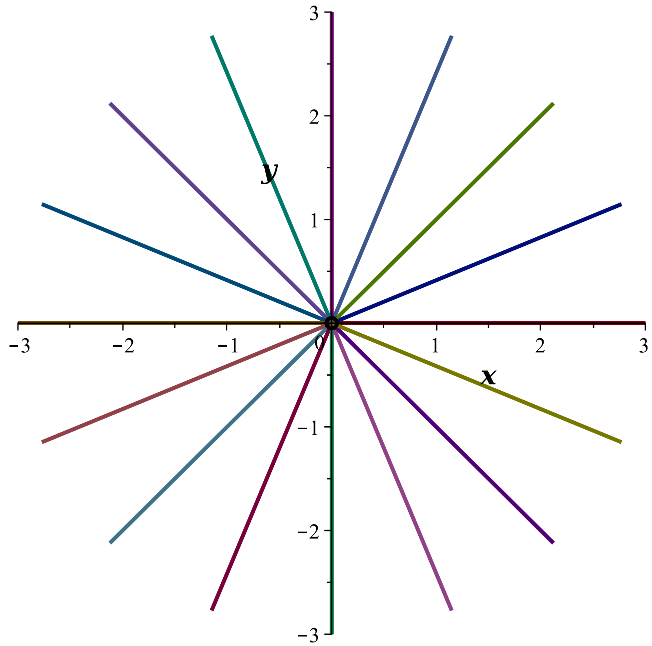
$$\alpha_1\,(t,(x,y)) = (x+ty,\ y)$$



$\mathcal{F} = \left\{y^3\right\}$ is separating on $\mathbb{R}^2\backslash\{(0,0)\}$,
but it does not generate $\mathbb{R}(\mathcal{Z})^{\mathcal{G}} = \mathbb{R}(y)$.

$\mathbb{R}^* \curvearrowright \mathbb{R}^2$:

$$\alpha_2(a,(x,y)) = \left(a^2\,x,\ a^2\,y\right)$$



$\mathcal{F} = \left\{\frac{y}{x}\right\}$ generates $\mathbb{R}(\mathcal{Z})^{\mathcal{G}}$, but is not generically separating.

(Note that $\mathbb{R}[\mathcal{Z}]^{\mathcal{G}} = \mathbb{R}$, so polynomial invariants do not separate any orbits)

What happens over $\mathbb{C}$?

Foundation:

- Basic definitions and examples
- Motivation and applications
- Smooth vs. algebraic setup
- Binary forms under linear changes of variables
- Planar curves under rigid motions
- History remarks

Cross-section method for computing invariants:

- Smooth construction
- Algebraic construction

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Binary forms under linear changes of variables

- $\mathcal{Z} = \mathcal{B}_m = \{$homogeneous polyn. over $\mathbb{C}$ in 2 variables of degree m $\}$
- $\mathcal{G} = SL_2(\mathbb{C}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \, a_{11}\, a_{22} - a_{12}\, a_{21} = 1 \right\}.$

The standrad action $\alpha_1 : \mathcal{G} \curvearrowright \mathbb{C}^2$ induces $\alpha_2 : \mathcal{G} \curvearrowright \mathcal{B}_m$:

$\boxed{(gf)(\mathbf{z}) = f(g^{-1}\mathbf{z})}$ for any $f \in \mathcal{B}_m, \mathbf{z} \in \mathbb{C}^2, \mathbf{g} \in \mathcal{G}$.

Example: For a binary quadratics: $f(x,y) = c_2 x^2 + 2c_1 xy + c_0 y^2$

$$(\mathbf{g}f)(x,y) = f(a_{11}x - a_{21}y, -a_{12}x + a_{22}y) = C_2 x^2 + 2C_1 xy + C_0 y^2,$$

where

$$\begin{aligned} C_2 &= {a_{21}}^2 c_0 - 2a_{21}a_{22}c_1 + {a_{22}}^2 c_2, \\ C_1 &= a_{11}a_{21}c_0 + (a_{11}a_{22} + a_{12}a_{21})c_1 - a_{12}a_{22}c_2 \\ C_0 &= {a_{11}}^2 c_0 - 2a_{11}a_{12}c_1 + {a_{12}}^2 c_2. \end{aligned}$$

Classical problem:

Terminology:

- invariants = invariant functions under the action $\alpha_2 : \mathcal{G} \curvearrowright \mathcal{B}_m$,
  (depend only on the coefficients.)

- covariants = invariant functions under the action
  $(\alpha_1 \times \alpha_2) : \mathcal{G} \curvearrowright \left(\mathbb{C}^2 \times \mathcal{B}_m\right)$,
  (depend on the coefficients and variables.)

Problem: find minimal generating sets for the rings of polynomial invariants and covariants and syzygies (relationships) between the generators.

Binary quadratics: $f = c_2x^2 + 2c_1xy + c_0y^2$.

- $\mathbb{C}[\mathcal{B}_2]^{SL_2(\mathbb{C})}$ is generated by $\Delta_2 = {c_1}^2 - c_0c_2$.

- $\mathbb{C}[\mathbb{C}^2 \times \mathcal{B}_2]^{SL_2(\mathbb{C})}$ is generated by Δ_2 and f.

Binary cubics: $f = c_3x^3 + 3c_2x^2y + 3c_1xy^2 + c_0y^3$

- $\mathbb{C}[\mathcal{B}_3]^{SL_2(\mathbb{C})}$ is generated by Δ_3.

- $\mathbb{C}[\mathbb{C}^2 \times \mathcal{B}_2]^{SL_2(\mathbb{C})}$ is generated by Δ_3, f, $H = f_{xx}f_{yy} - f_{xy}^2$ and $T = f_xH_y - f_yH_x$. The basis syzygy is:

$$T^2 = 2^4 3^6\,\Delta_3 f^2 - H^3$$

As the degree (and the number of variables) grow the problem becomes overwhelming!

Polynomial invariants and covariants:

- `https://en.wikipedia.org/wiki/Invariant_of_a_binary_form`

- R. Lercier and M. Olive *"Covariant algebra of the binary nonic and the binary decimic"*. Contemp. Math., (2017) (Table 1)

Rational invariants and covariants for binary forms:

- $\mathbb{C}(\mathcal{B}_m)^{SL_2}$ is generated by $m-2$ algebraically independent invariants

- $\mathbb{C}\left(\mathbb{C}^2 \times \mathcal{B}_m\right)^{SL_2}$ is generated by m algebraically independent covariants.

V. L. Popov and E. B. Vinberg. 1994. ”Invariant theory.” (Thm. 2.12)

T. Maeda, “On the invariant field of binary octavics”, Hiroshima Math. J. (1989)

The signature construction in the $\boxed{\text{third part}}$ of the tutorial gives a constructive solutions for solving equivalence problem for binary forms of any degree using just two rational covariants!

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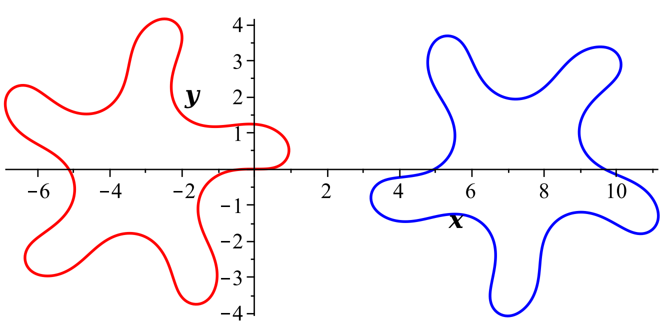
Smooth planar curves under rotations and translations

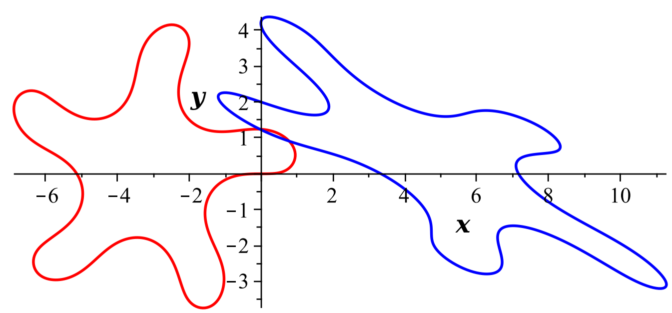
- $\mathcal{Z} = \{$smooth planar curves$\}$
- $\mathcal{G} = SE_2(\mathbb{R}) = SO_2(\mathbb{R}) \ltimes \mathbb{R}^2$

The standrad action $\alpha_1 : \mathcal{G} \curvearrowright \mathbb{R}^2$ induces $\alpha_2 : \mathcal{G} \curvearrowright \underline{\mathcal{Z}}$

(Other groups: full Euclidean $E_2(\mathbb{R})$, special affine $SA_2(\mathbb{R})$, affine $A_2(\mathbb{R})$, projective $PGL_2(\mathbb{R})$, are also of interest.)



$SE_2(\mathbb{R})$-congruent curves.



$SA_2(\mathbb{R})$-congruent curves.

What is a congruence criterion for planar curves under $SE_2(\mathbb{R})$?

Can we think of a canonical form for a curve which can be obtained by translations and rotations?

Can we think of a conical form for a curve with a selected point on it?

Curves congruence under $SE_2(\mathbb{R})$

Theorem: Let $\kappa_1 \colon \mathbb{R} \to \mathbb{R}$ and $\kappa_2 \colon \mathbb{R} \to \mathbb{R}$ be curvatures as functions of the arc-length s of curves Γ_1 and Γ_2. If there exists $c \in \mathbb{R}$, such that $\kappa_1(s) = \kappa_2(s + c)$ then Γ_1 and Γ_2 are congruent. The converse is true if Γ_1 and Γ_2 are simple.

Challenges in using this criterion:

- Arc-length parameterization.
- c-shift ambiguity.

These challenges can be overcome by the signature construction in the third part of the talk.

Euclidean curvature

- For Γ with a parameterization $\gamma(t) = (x(t), y(t))$

$$\kappa|_\Gamma(t) = \frac{\det(\gamma'(t), \gamma''(t))}{|\gamma'(t)|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

  and the arc-length one-form is $ds = |\gamma'|dt = \sqrt{x'(t)^2 + y'(t)^2}\,dt.$

- If Γ is the graph of $y = u(x)$, then

$$\kappa = \frac{u_{xx}}{\left(u_x^2 + 1\right)^{\frac{3}{2}}}.$$

  and the arc-length one-form is $ds = \sqrt{1 + u_x^2}\,dx.$

κ is a differential invariant.

κ be viewed as a “usual” invariant function from the jet space to $\mathbb{R}$.

Jet space and prolongation:

The k-th order jet space of planar curves $J^k = \left\{\left(x, y, y^{(1)}, \ldots, y^{(k)}\right)\right\}$.

Formally, $y^{(k)}$ is viewed as an independent coordinate function, but we define the prolongation of an action keeping in mind that $y^{(k)}$ is the "place holder" for the k-th derivative of y with respect to x:

Prolongation of $SE_2(\mathbb{R})$ action to J^2:

$$X = c\,x - s\,y + a, \quad Y = s\,x + c\,y + b,$$

$$Y^{(1)} = \frac{DY}{DX} = \frac{s + cy^{(1)}}{c - sy^{(1)}}, \quad Y^{(2)} = \frac{DY^{(1)}}{DX} = \frac{y^{(2)}}{(c - sy^{(1)})^3}.$$

$D = \frac{\partial}{\partial x} + y^{(1)}\frac{\partial}{\partial y} + y^{(2)}\frac{\partial}{\partial y^{(1)}} + \ldots$ is the total derivative w.r.t. x.

Invariants under the prolonged action on J^k are called differential invariants.

κ is an invariant function on J^1.

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Algebraic invariant theory

- The birth: A. Cayley (1845). *On theory of linear transformations*. Cambridge Math. J. inspired by G. Boole (1841) and (1842). *Exposition of a general theory of linear transformations*. Cambridge. Math. J.

- Main efforts: compute generating sets for polynomial invariants and covariants (also called integral bases) of n-ary forms (also called "quantics").

  - n-ary forms are homogeneous polynomials in n-variables.

  - Is there a finite generating set?

- A. Cayley (1855) *A second memoir upon quantics. Proceedings of Royal Society*: There is no finite generating set for the rings of polynomial invariants/covariants for binary forms of high degrees. Wrong!

- P. Gordan (1868) Crelle's Journal: Existence of finite generating sets for polynomial invariants/covariants for binary forms of any degree.

- D. Hilbert (1890) *Ueber die Theorie der algebraischen Formen*, Math. Annalen: Existence of finite generating sets for the rings of polynomial invariants/covariants of n-ary forms of any degree.
  - Hilbert's Basis Theorem and Syzygy Theorem appeared as lemmas in this paper!
  - There is an English translation of Hilbert's papers on invariants theory by Ackerman with comments by Hermann (1978).

Hermann Weyl, “Invariants”, (1939):

“The theory of invariants came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley’s Jovian head. Her Athens over which she ruled and which she served as a tutelary and beneficent goddness was projective geometry. From the beginning she was dedicated to a proposition that all projective coordinate systems are created equal. ”

Joseph Kung and Gian-Carlo Rota, “The Invariants Theory of Binary Forms”. (1984)

“Like the Arabian phoenix rising out of its ashes, the theory of invariants, pronounced dead at the turn of the century, is once again at the forefront of mathematics. During its long eclipse, the language of modern algebra was developed, a sharp tool now at long last being applied to the very purpose for which it was intended”.

A booklist recommendation for the modern state of the field with connection to the classics:

- Jean Dieudonné and James B. Carrell, “Invariant Theory Old and New” (1971)

- Vladimir Popov and Ernest Vinberg, “Invariant Theory” (1989 in Russian, 1994 in English)

- Bernd Sturmfels, “ Algorithms in Invariant Theory” (1993, 2008)

- Harm Derksen and Gregor Kemper, “Computational invariant theory” (2002, 2015)

- Peter Olver, “Classical invariant theory” (1999)

Differential invariant theory

- S. Lie,
  - "Theorie der Transformations-gruppen" Math Annalen. (1880):
    Infinitesimal transformations $\rightarrow$ Lie algebras.
  - "Über Differentialinvarianten" Math Annalen. (1884):
    Differential invariants.

  (English translation by Ackerman with comments by Hermann (1975-1976).)

  Motivation: to extend Galois theory to differential equations.

- A. Tresse, "Sur les invariants defférentiels des group continus de transformations", Acta Math (1894):
  Differential algebra of differential invariants is finitely generated.

- É. Cartan,
  - “La méthode du repère mobile, la théorie des groupes continus, et les espaces généralisés ”, Actualités Scientifiques et Industrielle (1935):

    The moving frame method to compute differential invariants.
  - “Les problèmes d’équivalence”. Gauthier-Villars, Paris. (1953):

    Equivalence problems can be solved by studying relationships between differential invariants.

Sophus Lie, “Geometrie der Berührungstransformationen.” (1896)

“... in our century, mathematics has split up into many very extensive areas, and this division has often lead the representatives of one area misjudge the importance of others, so that, to detriment of their own discipline, fruitful ideas from the outside has been ignored”

A booklist recommendation for the modern state of the field with connection to the classics:

- Peter Olver, “Equivalence, Invariants and Symmetry” (1995)

- Jeanne Clelland,“From Frenet to Cartan : The Method of Moving Frames” (2017)

- Thomas Ivey and J.M. Landsberg, “Cartan for beginners: differential geometry via moving frames and exterior differential systems” (2003)

- Peter Hydon, “Symmetry Methods for Differential Equations” (2000)

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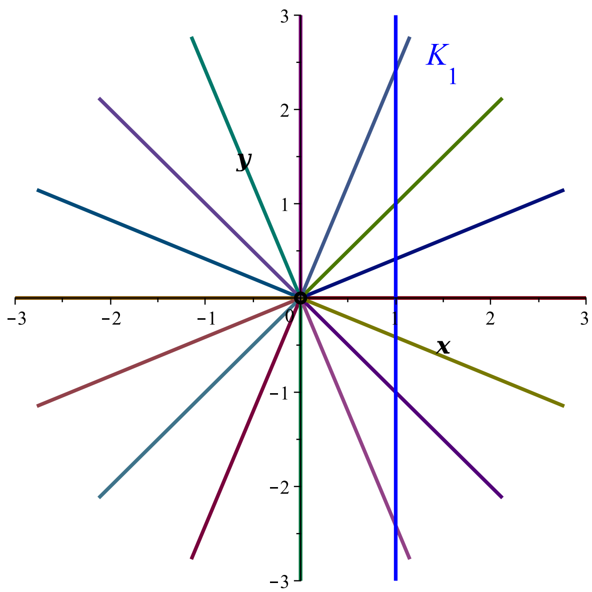
Smooth cross-section

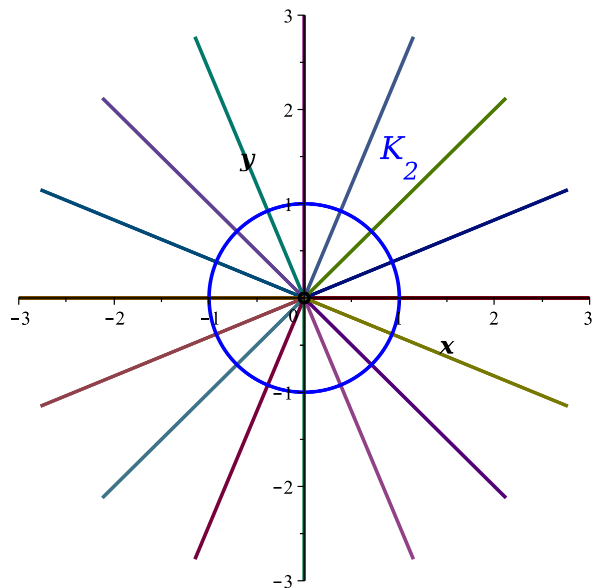
A submanifold $\mathcal{K} \subset \mathcal{Z}$ is called a local cross-section for a (local) action $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}$ if there exists $\mathcal{U} \underset{open}{\subset} \mathcal{Z}$, called the domain of the cross-section, s.t, $\forall \mathbf{z} \in \mathcal{U}$, the connected component $\mathcal{O}^0_{\mathbf{z}}$ of $\mathcal{O}_{\mathbf{z}} \cap \mathcal{U}$, containing $\mathbf{z}$, intersects $\mathcal{K}$ transversally at a single point:

$$\mathcal{O}^0_{\mathbf{z}} \cap \mathcal{K} = \{\tilde{\mathbf{z}}\} \text{ and } T\mathcal{K}|_{\tilde{\mathbf{z}}} \oplus T\mathcal{O}_{\mathbf{z}}|_{\tilde{\mathbf{z}}} = T\mathcal{Z}|_{\tilde{\mathbf{z}}}.$$

Example: Two cross-sections for $\alpha(t,(x,y)) = \left(e^t\, x,\, e^t\, y\right)$ and their domains.



$\mathcal{U}_1 = \{(x,y) | x > 0\}$.



$\mathcal{U}_2 = \mathbb{R}^2 \backslash \{(0,0)\}$.

Cross-sections are easy to find!

If s is the maximal orbit dimension, then in a neighborhood of a generic point $\mathbf{z}$:

- A generic manifold of co-dimension s is a cross-section.

- There exists a choice of s coordinate functions and constants such that equations:

$$z_{i_1} = c_1, \ldots, z_{i_s} = c_s$$

  define a cross-section.

Invariantization: If $\mathcal{K}$ is a cross-section on $\mathcal{U}$, then

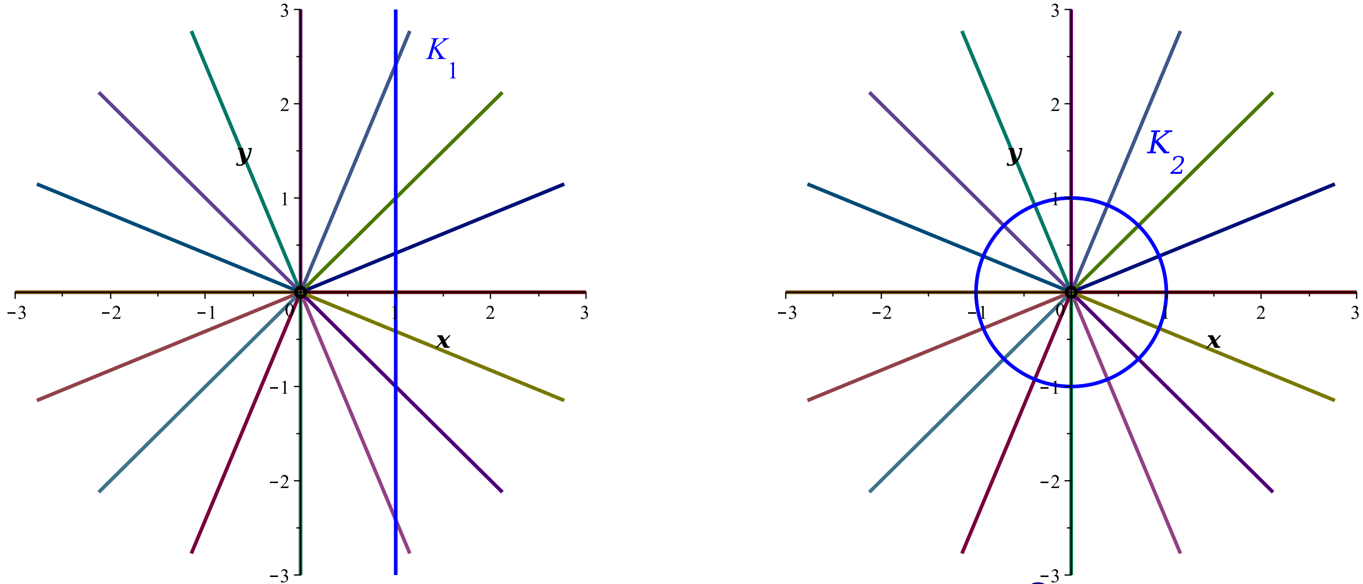
- $\forall f \in C^\infty(\mathcal{U}) = \{$smooth functions on $\mathcal{U}\}$
  $\exists! \iota f \in C^\infty(\mathcal{U})^{\mathcal{G}} = \{$locally invariant smooth functions on $\mathcal{U}\}$, s.t.

$$\iota f|_{\mathcal{K}} = f|_{\mathcal{K}}$$

- Let $z = (z_1, \ldots, z_n)$ be coordinate functions on $\mathcal{U}$.
  Then $\iota z = (\iota z_1, \ldots, \iota z_n)$ are called normalized invariants.
  - Explicitly, ιz are obtained by expressing the coordinates of the intersection point $\tilde{\mathbf{z}} = \mathcal{O}^0_{\mathbf{z}} \cap \mathcal{K}$ in terms of coordinates of $\mathbf{z}$.

    Not that easy to do. One of our goals is to show how.
  - The set ιz separates connected components of the orbits on $\mathcal{U}$.
  - The set ιz has replacement property: $\iota f(z) = f(\iota z)$.

Example: Two cross-sections for $\alpha(t,(x,y)) = \left(e^t x,\, e^t y\right)$.



$\mathcal{U}_1 = \{(x,y) | x > 0\}$.
$\mathcal{K}_1$-normalized invariants:
$(\iota x, \iota y) = \left(1, \frac{y}{x}\right)$

$\mathcal{U}_2 = \mathbb{R}^2 \backslash \{(0,0)\}$.
$\mathcal{K}_2$-normalized invariants:
$(\tilde{\iota} x, \tilde{\iota} y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

- Normalized invariants satisfy the equations of the corresponding cross-section: $\iota x = 1$ and $(\tilde{\iota} x)^2 + (\tilde{\iota} y)^2 = 1$.

Theorem: For a local action $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}$ and a generic $\mathbf{z} \in \mathcal{Z}$ their exists an open neighborhood $\mathcal{U}$ such that:

- There exists a cross-section $\mathcal{K}$, with a domain $\mathcal{U}$, equal to the zero set of $s = \dim \mathcal{O}_{\mathbf{z}}$ independent smooth functions $K_1(z), \ldots, K_s(z) \in C^\infty(\mathcal{U})$.

- The normalized invariants $\iota z = (\iota z_1, \ldots, \iota z_n)$ (where $z = (z_1, \ldots, z_n)$ are coordinate functions on $\mathcal{U}$) generate $C^\infty(\mathcal{U})^{\mathcal{G}}$ (due to their replacement property).

- $K_1(\iota z) = 0, \ldots, K_s(\iota z) = 0$ is a set of independent relationships (syzygies) among normalized invariants, generating all such relationships.

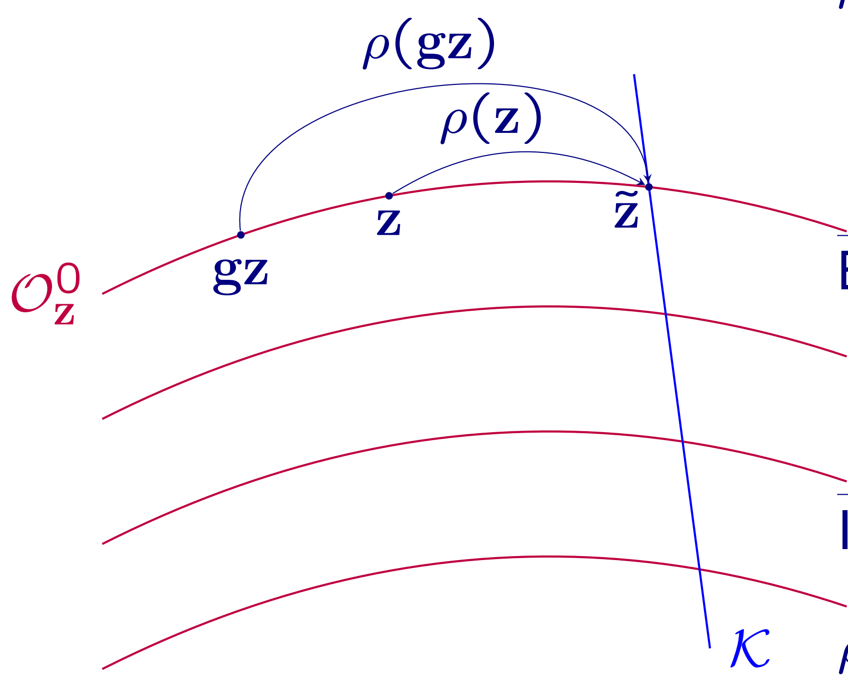
- There exists a subset of $n - s$ independent normalized invariants $\iota z_{i_1}, \ldots, \iota z_{i_s}$ generating $C^\infty(\mathcal{U})^{\mathcal{G}}$ (a fundamental set).

Here independent means functionally independent.

Warning: $\mathcal{U}$ can be very small (metric topology). The invariants are local.

Moving frame map

M. Fels and P. J. Olver, " Moving Coframes.II. Regularization and Theoretical Foundations". Acta Appl. Math. (1999).



$\rho\colon \mathcal{U} \to \mathcal{G}$ is a smooth map s.t.

- $\rho(\mathbf{z}) = e, \quad \forall \mathbf{z} \in \mathcal{K}.$
- $\rho(\mathbf{z})\mathbf{z} = \tilde{\mathbf{z}} = \mathcal{O}^0_{\mathbf{z}} \cap \mathcal{K}, \ \forall \mathbf{z} \in \mathcal{U},$

By construction

$$\rho(\mathbf{gz})(\mathbf{gz}) = \rho(\mathbf{z})\,\mathbf{z}.$$

If the action is free

$\rho(\mathbf{gz}) = \rho(\mathbf{z})\cdot\mathbf{g}^{-1}$ (right equivariance)

$\rho(\mathbf{z})$ is a group element that "brings" $\mathbf{z}$ to $\mathcal{K}$ along the connected piece of the orbit.

The coordinate components of $\rho(\mathbf{z})\mathbf{z}$ are normalized invariants.

Example 1: $\alpha(t,(x,y)) = \Big(e^t\,(x\cos t - y\sin t)\,,\; e^t\,(x\sin t + y\cos t)\Big)$

- $\mathcal{K} = \{(x,y)\,|\,x^2+y^2=1\}$ is a cross-section.
- $\rho(x,y)$ is a group parameter that brings (x,y) to $\mathcal{K}$. To find $\rho(x,y)$ we solve for t the equation:

$$\Big(e^t\,(x\cos t - y\sin t)\Big)^2 + \Big(e^t\,(x\sin t + y\cos t)\Big)^2 = 1$$

$$t = -\ln r = \rho(x,y), \text{where } r = \sqrt{x^2+y^2}.$$

- Normalized invariants are obtained by substituting $-\ln r$ for t in the equations of the action:

$$\iota x = \frac{1}{r}\left(x\cos(\ln r) + y\sin(\ln r)\right),$$
$$\iota y = \frac{1}{r}\left(y\cos(\ln r) - x\sin(\ln r)\right).$$

Repeat this procedure for a cross-section $\widetilde{\mathcal{K}} = \{(0,y)\,|y>0\}$ with the domain $\widetilde{\mathcal{U}} = \{(x,y)\,|y>0\}$. Compare the normalized invariants.

Example 2: $SE_2(\mathbb{R})$ action on the <u>third</u> order jet space J^3 of curves.

$$X = c\,x - s\,y + a, \quad Y = s\,x + c\,y + b, \quad c^2 + s^2 = 1,$$

$$Y^{(1)} = \frac{s + cy^{(1)}}{c - sy^{(1)}}, \quad Y^{(2)} = \frac{y^{(2)}}{(c - sy^{(1)})^3},$$

$$Y^{(3)} = \frac{\left(-sy^{(1)}y^{(3)} + 3s(y^{(2)})^2 + cy^{(3)}\right)}{\left(c - sy^{(1)}\right)^5}.$$

- Cross-section on J^3 is defined by: $x = 0,\ y = 0,\ y^{(1)} = 0$.

- To find $\rho\colon\ J^3 \to SE_2(\mathbb{R})$ we solve for c, s, a, b the equations:

$$X = 0, \quad Y = 0, \quad Y^{(1)} = 0, \quad c^2 + s^2 = 1 :$$

$$c = +/- \frac{1}{\sqrt{\left(y^{(1)}\right)^2 + 1}},\ s = -/+ \frac{y^{(1)}}{\sqrt{\left(y^{(1)}\right)^2 + 1}}$$

$$a = \frac{-yy^{(1)} - x}{\sqrt{\left(y^{(1)}\right)^2 + 1}},\ b = \frac{xy^{(1)} - y}{\sqrt{\left(y^{(1)}\right)^2 + 1}}.$$

Choose signs so that $\rho|_{\mathcal{K}} = e$,

(i.e. $c = 1,\ s = 0,\ a = 0,\ b = 0$, when $x = y = y^{(1)} = 0$.)

- Normalized invariants are obtained by substituting $\rho(x, y, y^{(1)})$ for c, s, a, b in the equations of the action:

$$\iota x = 0, \quad \iota y = 0, \quad \iota y^{(1)} = 0,$$

$$\iota y^{(2)} = \frac{y^{(2)}}{\left(\left(y^{(1)}\right)^2 + 1\right)^{\frac{3}{2}}} = \frac{y_{xx}}{\left(y_x^2 + 1\right)^{\frac{3}{2}}} := \kappa,$$

$$\iota y^{(3)} = \frac{\left(1 + y_x^2\right) y_{xxx} - 3\, y_x\, y_{xx}^2}{\left(1 + y_x^2\right)^3} := \kappa_s.$$

(The Euclidean curvature κ and its derivative κ_s with respect to the arc-length $ds = \sqrt{y_x^2 + 1}$.)

- $\mathrm{g} \in SE_2(\mathbb{R})$ with $c = -1, s = 0, a = 0, b = 0$ sends
$y^{(1)} \to y^{(1)}, \qquad y^{(2)} \to -y^{(2)}, \qquad y^{(3)} \to y^{(3)}$ and so

$$\kappa \to -\kappa, \qquad \kappa_s \to \kappa_s.$$

  - κ is a local invariant (it is invariants when g is "sufficiently" close to e).
  - κ_s is a global rational invariant.

Steps: Let the maximal orbit dimension for $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}$ be s.

i. Write α in terms of coordinates on $\mathcal{Z}$ and group parameters:

$$Z = \alpha(\lambda, z), \qquad (*)$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $z = (z_1, \ldots, z_n)$, $Z = (Z_1, \ldots, Z_n)$.

ii. Near $\mathbf{z} \in \mathcal{Z}$ such that $\dim \mathcal{O}_{\mathbf{z}} = s$, choose a cross-section $\mathcal{K} \subset \mathcal{U} \subset \mathcal{Z}$ defined by a set of s equations on $\mathcal{U}$:

$$K = \{K_1(z) = 0, \ldots, K_s(z) = 0\}.$$

iii. Substitute $(*)$ into K:

$$\{K_1(\alpha(\lambda, z)) = 0, \ldots, K_s(\alpha(\lambda, z)) = 0\}.$$

and solve for group parameters $\Longrightarrow$ a moving frame map

$$\lambda = \rho(z). \qquad (**)$$

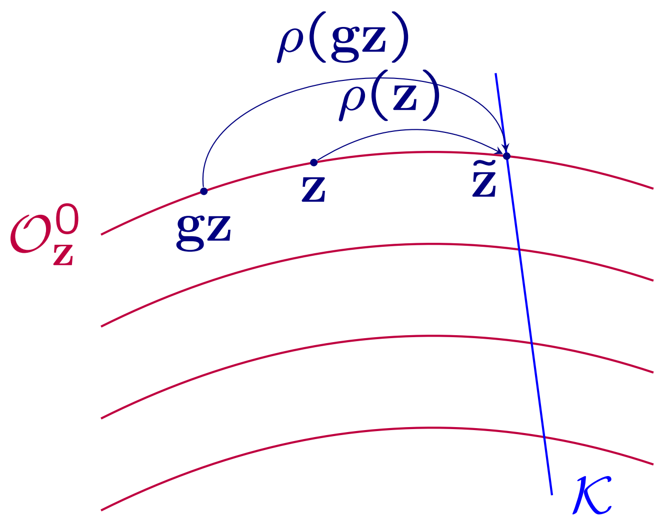
iv. Substitute $(**)$ into $(*)$ $\Longrightarrow$ normalized invariants: $\iota z = \alpha\left(\rho(z), z\right)$.

In the algebraic context: (solve and substitute) = eliminate.

Why ρ is called a moving frame map?

For the $SE_2(\mathbb{R}) = SO_2(\mathbb{R}) \ltimes \mathbb{R}^2$ action on the planar curves:

Fels-Olver moving frame map:



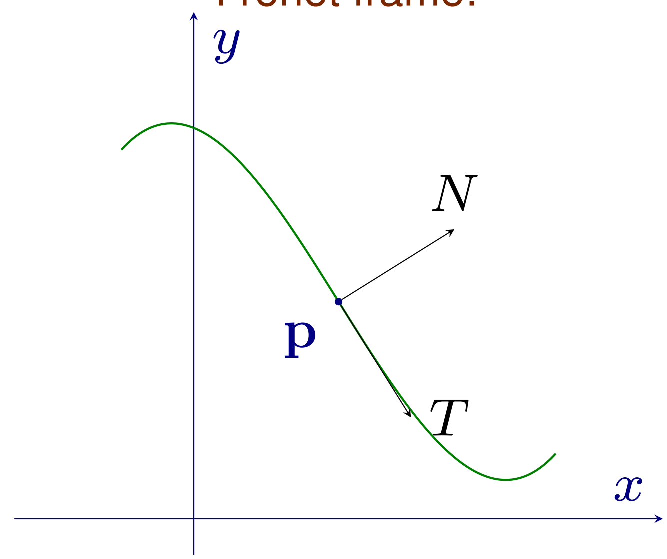
We find $\rho\colon J^1 \to SE_2(\mathbb{R})$,

$$\rho\left(x, y, y^{(1)}\right) = \left(\begin{bmatrix} c & -s \\ s & c \end{bmatrix}, (a, b)\right).$$

$$c = \frac{1}{\sqrt{\left(y^{(1)}\right)^2 + 1}}, \quad s = -\frac{y^{(1)}}{\sqrt{\left(y^{(1)}\right)^2 + 1}}$$

$$a = \frac{-yy^{(1)} - x}{\sqrt{\left(y^{(1)}\right)^2 + 1}}, \quad b = \frac{xy^{(1)} - y}{\sqrt{\left(y^{(1)}\right)^2 + 1}}.$$

Frenet frame:



$$T = \begin{bmatrix} \frac{1}{\sqrt{1+y^{(1)2}}} \\ \frac{y^{(1)}}{\sqrt{1+y^{(1)2}}} \end{bmatrix}, \quad N = \begin{bmatrix} -\frac{y^{(1)}}{\sqrt{1+y^{(1)2}}} \\ \frac{1}{\sqrt{1+y^{(1)2}}} \end{bmatrix}.$$

$$\tilde{\rho}\left(x, y, y^{(1)}\right) = ([T, N], (x, y))$$

is a map $J^1 \to SE_2(\mathbb{R})$.

$$\tilde{\rho}(\mathbf{z}) = \rho(\mathbf{z})^{-1}.$$

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- Basic definitions and examples
- Motivation and applications
- Smooth vs. algebraic setup
- Binary forms under linear changes of variables
- Planar curves under rigid motions
- History remarks

Cross-section method for computing invariants:

- Smooth construction
- Algebraic construction

Differential signature construction for solving the equivalence problem:

- Planar curves
- Binary forms

Action:

- $\mathbb{K}$ is a field of characteristic 0 with the algebraic closure $\overline{\mathbb{K}}$.

- A group $\mathcal{G} \subset \mathbb{K}^\ell$ is the zero set of a radical equi-dimensional ideal
$$G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_\ell]$$
and is a restriction (a set of $\mathbb{K}$-points) of an affine group $\widehat{\mathcal{G}}$ defined over $\overline{\mathbb{K}}$.

- $\mathcal{Z} = \mathbb{K}^n$ and $\widehat{\mathcal{Z}} = \overline{\mathbb{K}}^n$.

- The action $\alpha\colon \mathcal{G} \curvearrowright \mathcal{Z}$ is a restriction of a rational action $\widehat{\alpha}\colon \widehat{\mathcal{G}} \curvearrowright \widehat{\mathcal{Z}}$, given, in coordinates, by n rational functions over $\mathbb{K}$:
$$Z_i = \frac{\gamma_i(\lambda, z)}{\delta_i(\lambda, z)}, \quad i = 1, \ldots, n,$$
where both $z = (z_1, \ldots, z_n)$ and $Z = (Z_1, \ldots, Z_n)$ are coordinate functions on $\widehat{\mathcal{Z}}$, called the source and the target coordinates, respectively.

- Action ideal: ($\delta = LCM\{\delta_1, \ldots, \delta_n\}$ and μ is a “dummy variable”.)
$$A = \langle \{\delta_i(\lambda, z) Z_i - \gamma_i(\lambda, z)\}_{i=1}^n\,,\ \delta(\lambda, z)\mu - 1\rangle + G \subset \mathbb{K}[\lambda, \mu, z, Z].$$

Algebraic cross-section

- A subvariety $\mathcal{K} \subset \mathcal{Z}$ defined by an ideal $K \subset \mathbb{K}[\mathcal{Z}]$ is a cross-section of degree d for an action $\alpha\colon \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$ if the corresponding subvariety $\widehat{\mathcal{K}} \subset \widehat{\mathcal{Z}}$ defined by extending K to $\overline{\mathbb{K}}[\widehat{\mathcal{Z}}]$ is irreducible and intersects each generic orbit of the action $\widehat{\alpha}\colon \widehat{\mathcal{G}} \times \widehat{\mathcal{Z}} \to \widehat{\mathcal{Z}}$ at exactly d points.

- If $d = 1$, the cross-section is called rational.

Cross-sections exist and easy to find, but a rational cross-section does not always exist!

- A generic irreducible variety of co-dimension s is a cross-section.

- There exists a choice of s coordinate functions and constants such that equations:

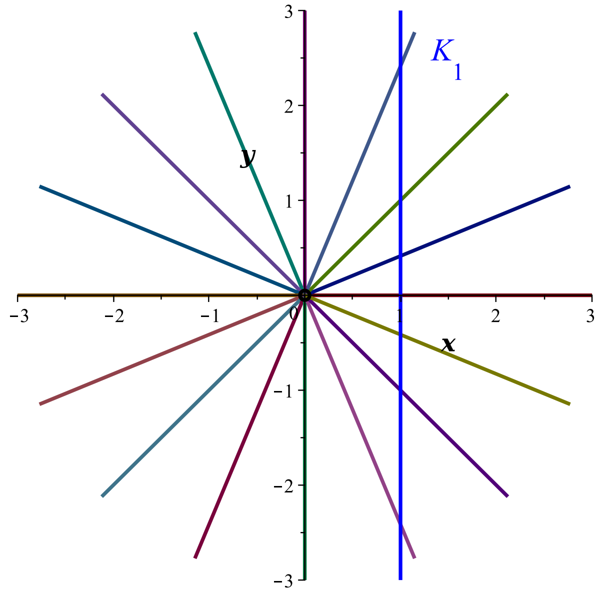
$$z_{i_1} = c_1, \ldots, z_{i_s} = c_s$$

define a cross-section.

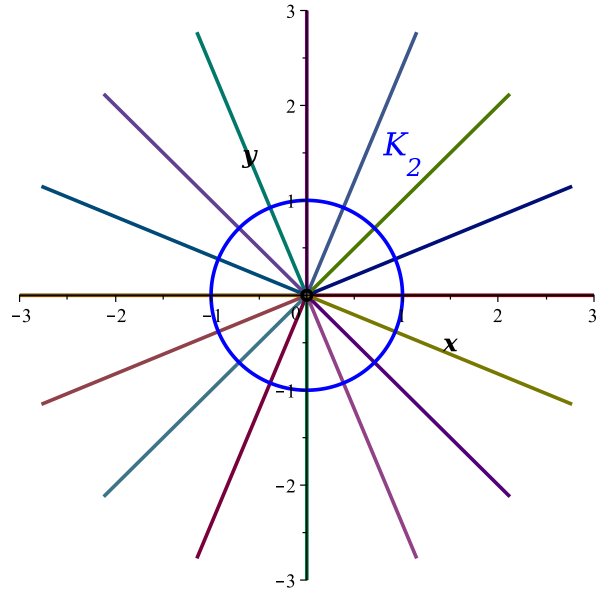
Why do we need to define the c.s. over the algebraic closure?

For positive scaling $\alpha\colon \mathcal{G} = \mathbb{R}^* \curvearrowright \mathcal{Z} = \mathbb{R}^2$: $\alpha(a, x, y) = (a^2 x, a^2 y)$, since we define the generic number of points in $\mathcal{K} \cap \mathcal{O}_{\mathbf{z}}$ over $\overline{\mathbb{C}}$, then

$\mathcal{K}_1$ is a cross-section of degree 1:



$\mathcal{K}_2$ is a cross-section of degree 2:



What would happen if we made the definition over $\mathbb{R}$?

Action $+$ cross-section

$$A \subset \mathbb{K}[\lambda, \mu, z, Z]$$ the action ideal

$$+$$

$$K \subset \mathbb{K}[Z]$$ the cross-section ideal written in terms of target variables

$$\Downarrow$$

$$I = (A + K) \cap \mathbb{K}[z, Z]$$ the graph-section ideal

- The variety of I in $\widehat{Z} \times \widehat{Z}$ is the closure of the set:

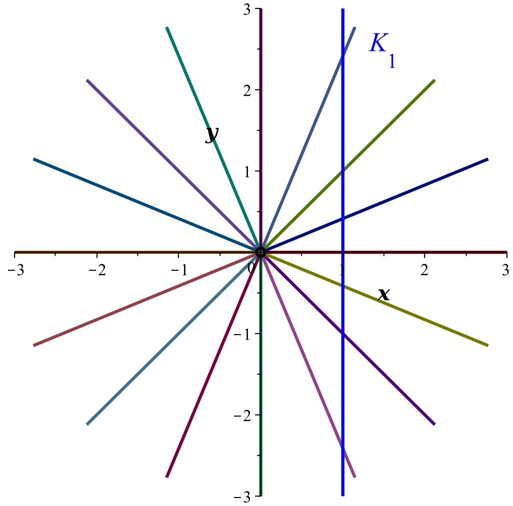
$$\mathcal{I} = \{(\mathbf{z}, \tilde{\mathbf{z}}) \in \widehat{\mathcal{Z}} \times \widehat{\mathcal{K}} \mid \exists \mathbf{g} \in \widehat{\mathcal{G}}, \text{ s.t. } \tilde{\mathbf{z}} = \widehat{\alpha}(\mathbf{g}, \mathbf{z})\}.$$

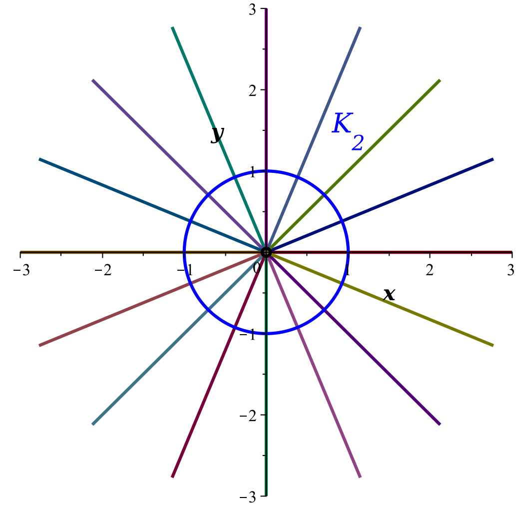
- I can be obtained by elimination of λ and μ from the ideal $A + K \subset \mathbb{K}[\lambda, \mu, z, Z]$

- We will use ideal $I^e \subset \mathbb{K}(z)[Z]$ generated by I in the ring $\mathbb{K}(z)[Z]$.

For positive scaling $\alpha : \mathcal{G} = \mathbb{R}^* \curvearrowright \mathcal{Z} = \mathbb{R}^2$: $\alpha(a, x, y) = (a^2 x, a^2 y)$

$$A = \left\langle X - a^2 x, \quad Y - a^2 y, \quad ab - 1 \right\rangle .$$





$$A + K_1 = A + < X - 1 >, \qquad A + K_2 = A + < X^2 + Y^2 - 1 >$$

$$I_1 = (A + K_1) \cap \mathbb{R}[x, y, X, Y] = \langle Y\,x - y,\ X - 1 \rangle ,$$

$$I_2 = (A + K_2) \cap \mathbb{R}[x, y, X, Y] = \left\langle X\,y - Y\,x, (x^2 + y^2)\,Y^2 - y^2 \right\rangle ,$$

$$I_1^e = \left\langle X - 1, Y - \frac{y}{x} \right\rangle . \qquad I_2^e = \left\langle Y - \frac{y}{x} X,\ X^2 - \frac{x^2}{x^2 + y^2} \right\rangle .$$

The coefficients are rational invariants! Is it true in general? Yes!

Properties of $I^e \subset \mathbb{K}(z)[Z]$

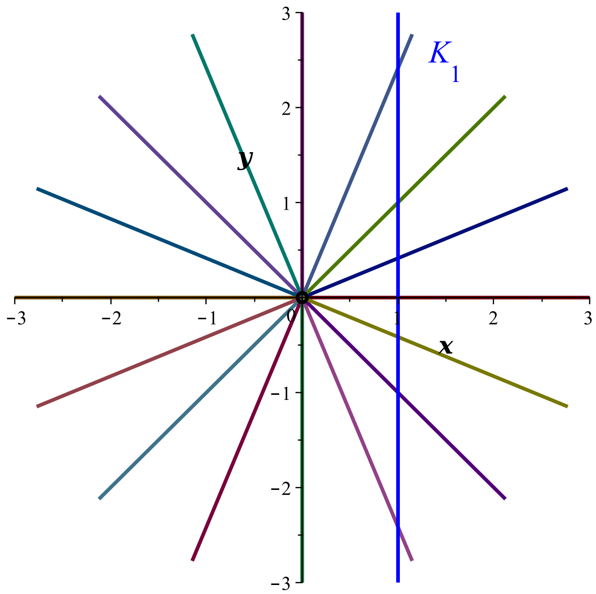
- For a generic $\mathbf{z}$, the specialization $I_{\mathbf{z}} \subset \overline{\mathbb{K}}[Z]$ of I^e is the ideal of the finite set $\widehat{\mathcal{O}_{\mathbf{z}}} \cap \widehat{\mathcal{K}}$ of cardinality d, and so I^e is zero-dimensional with

$$d = \mathrm{dim}(\overline{\mathbb{K}}(z)[Z] \backslash I^e).$$

- $\mathcal{I}_{\mathbf{gz}} = \mathcal{I}_{\mathbf{z}}$ for any $\mathbf{z} \in \mathcal{Z}$ and $\mathbf{g} \in \mathcal{G}$, and so the coefficients of a canonical generating set of I^e are invariant under the action of $\mathcal{G}$.

For positive scaling α: $\mathcal{G} = \mathbb{R}^* \curvearrowright \mathcal{Z} = \mathbb{R}^2$: $\alpha(a, x, y) = (a^2 x, a^2 y)$.

Cross-section of degree 1:

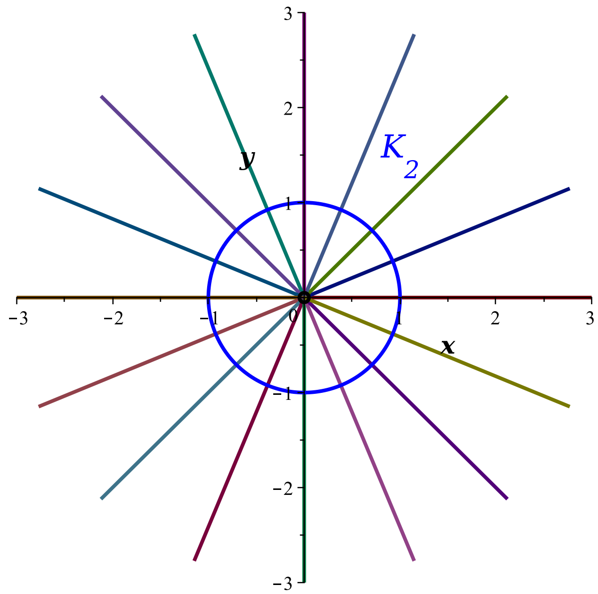


$$I_1^e = \left\langle X - 1, Y - \frac{y}{x} \right\rangle.$$

$\Downarrow$

minimal generating set $\left\{\frac{y}{x}\right\}$ of $\mathbb{R}(\mathcal{Z})^{\mathcal{G}}$.

Cross-section of degree 2:



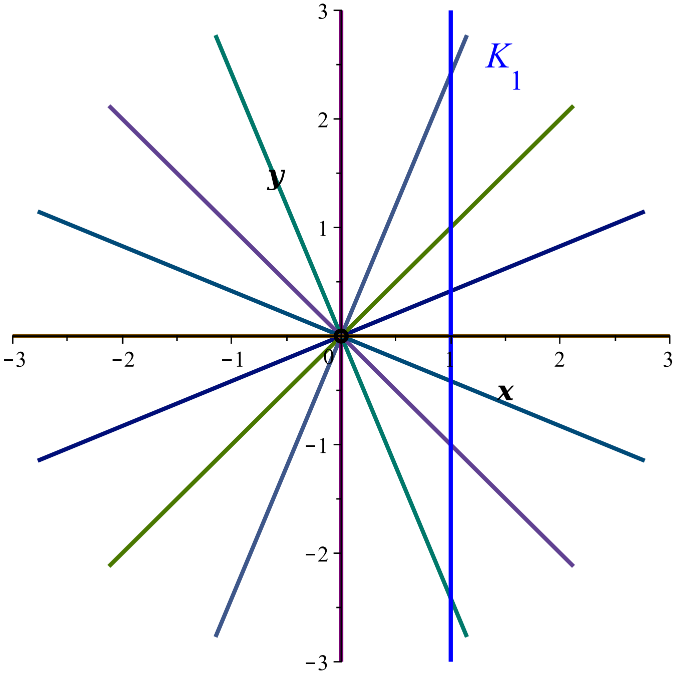
$$I_2^e = \left\langle Y - \frac{y}{x} X,\ X^2 - \frac{x^2}{x^2 + y^2} \right\rangle.$$

$\Downarrow$

non-minimal generating set $\left\{\frac{y}{x}, \frac{x^2}{x^2+y^2}\right\}$ of $\mathbb{R}(\mathcal{Z})^{\mathcal{G}}$.

We get the same generators for full scaling $\alpha(a,x,y) = (ax, ay)$ over $\mathbb{R}$ or $\mathbb{C}$:

Cross-section of degree 1:

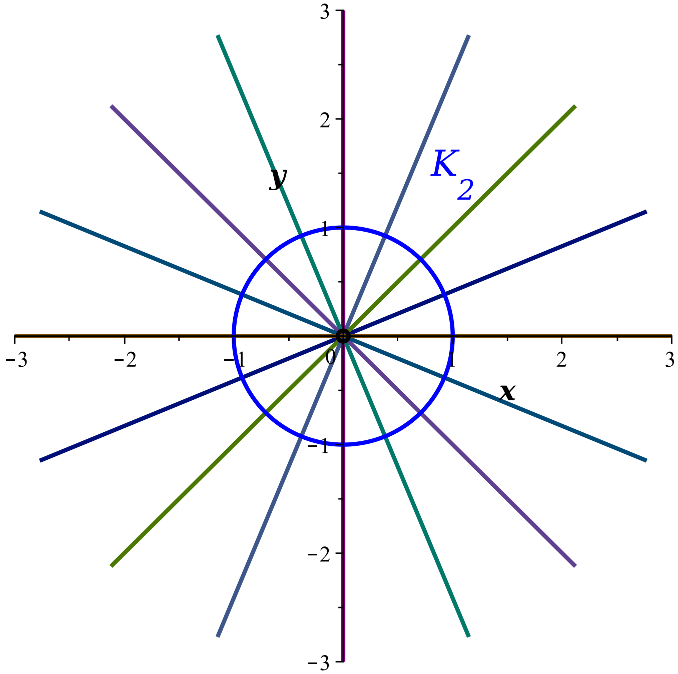


$$I_1^e = \left\langle X - 1, Y - \frac{y}{x} \right\rangle.$$

$\Downarrow$

minimal generating set $\{\frac{y}{x}\}$ of $\mathbb{R}(\mathcal{Z})^{\mathcal{G}}$.

Cross-section of degree 2:



$$I_2^e = \left\langle Y - \frac{y}{x} X,\ X^2 - \frac{x^2}{x^2 + y^2} \right\rangle.$$

$\Downarrow$

non-minimal generating set $\{\frac{y}{x}, \frac{x^2}{x^2+y^2}\}$ of $\mathbb{R}(\mathcal{Z})^{\mathcal{G}}$.

Graph-section theorem: E. Hubert and I. A. Kogan. (2007)

- "Rational invariants of a group action. Construction and rewriting." J. Symbolic Comput.
- "Smooth and algebraic invariants of a group action: local and global constructions." Found. Comput. Math.

- The coefficients of the reduced Gröbner basis of the graph-section ideal $I^e \subset \mathbb{K}(z)[Z]$, relative to any monomial order on the variables Z, is a generating set of the field $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ of rational invariants.

- $\mathbb{K}(\mathcal{K})$ is an algebraic extension of degree d of the invariant field $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$.

- Components of zeros of I^e belong to $\overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$ and have the same replacement properties as normalized invariants.

- If $\mathcal{K}$ is rational ($d = 1$), then

  i. $\mathbb{K}(\mathcal{Z})^{\mathcal{G}} \cong \mathbb{K}(\mathcal{K})$,

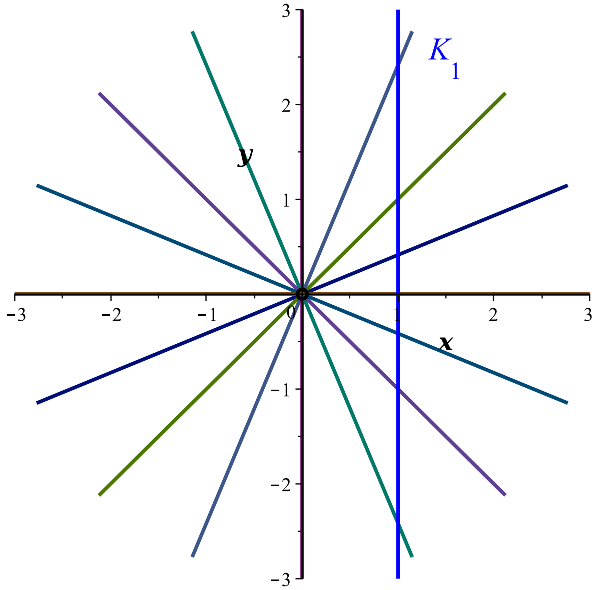
  ii. $Q = \{Z_i - r_i(z)\}_{i=1}^{n}$,

  iii. rational invariants $r_1, \ldots, r_n$ have the replacement property:

$$f(z) = f(r_1, \ldots, r_n) \text{ for any } f(z) \in \mathbb{K}(\mathcal{Z})^{\mathcal{G}}.$$

Example: full scaling $\alpha : \mathcal{G} = \mathbb{R}^* \curvearrowright \mathcal{Z} = \mathbb{R}^2$: $\alpha(a,x,y) = (ax, ay)$.

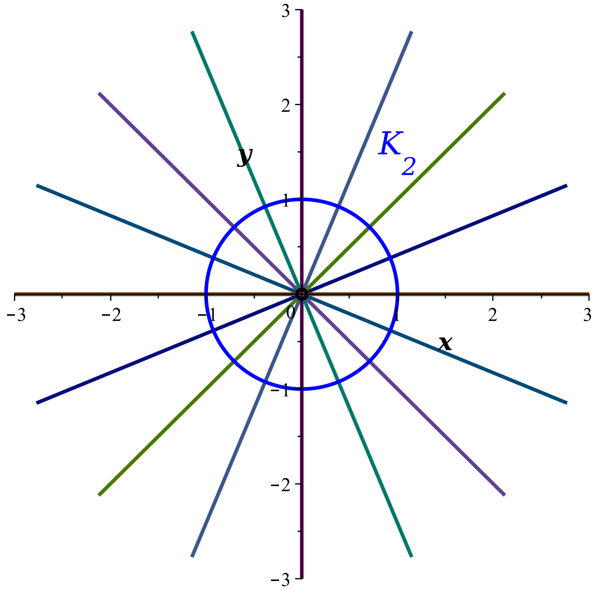
Cross-section of degree 1:



$$I_1^e = \left\langle X - 1, Y - \frac{y}{x} \right\rangle.$$

Components of the unique zero $\xi = \left(1, \frac{y}{x}\right)$ have replacement properties.

Cross-section of degree 2:



$$I_2^e = \left\langle Y - \frac{y}{x} X,\; X^2 - \frac{x^2}{x^2+y^2} \right\rangle.$$

Components of each of the two zeros:

$\xi_1 = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ and

$\xi_2 = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right)$

have replacement properties.

Previous and subsequent work

- Ideal $O = A \cap \mathbb{K}[z, Z]$ (no cross-section) (known as the graph-of-the-action ideal and Derksen ideal), or rather its extension $O^e \subset \mathbb{K}(z)[Z]$, has been used to obtain generating sets of the rational invariant field:
  - M. Rosenlicht, "Some basic theorems on algebraic groups". Amer. J. Math. (1956): The coefficients of the Chow form of O^e form a generating set of rational invariants.
  - V. L. Popov and E. B. Vinberg. "Invariant theory" (1994): If the coefficients of a generating set of O^e are invariant then they generate $\mathbb{K}(z)^{\mathcal{G}}$.
  - J. Müller-Quade and T. Beth, "Calculating generators for invariant fields of linear algebraic groups". Lecture Notes in Computer Science (1999) and J. Müller-Quade and R. Steinwandt. "Basic algorithms for rational function fields". J. Symbolic Comput. (1999): Coefficients of the reduced Gröbner basis of O^e generate $\mathbb{K}(z)^{\mathcal{G}}$. A procedure to express any invariant in terms of these coefficients is proposed for a linear algebraic group acting linearly on a finite-dimensional vector space.

- E. Hubert and I. A. Kogan. “Rational invariants of a group action. Construction and rewriting.” J. Symbolic Comput. (2007): Coefficients of the reduced Gröbner basis of O^e generate $\mathbb{K}(z)^{\mathcal{G}}$. A simple procedure, based on normal forms computation, to express any invariant in terms of these coefficients is proposed for any rational action.

- G. Kemper. “The computation of invariant fields and a new proof of a theorem by Rosenlicht.” Transformation Groups, (2007): a generalization of an the algorithm proposed by Müller-Quade and T. Beth to non-linear actions.

• Ideal O and a Reynolds operator can be used to compute a generating set of polynomial invariant ring (H. Derksen. “Computation of invariants for reductive groups.” Adv. Math. (1999))

• The notion of the cross-section can be generalized, at least for polynomial actions (G. Kemper. “Using extended Derksen ideals in computational invariant theory.” J. Symbolic Comput. (2016)).

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Cross-section method for computing invariants:

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Differential signature construction for solving the equivalence problem:

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Signature solution for the $SE_2(\mathbb{R})$-equivalence problem for planar curves:

Problem: Given two planar curves Γ_1 and Γ_2, decide whether or not they can be matched by a composition of rotations and translations?

On the third order jet space J^3 we found two non-constant normalized invariants:

$$\iota y^{(2)} = \frac{y^{(2)}}{\left(\left(y^{(1)}\right)^2 + 1\right)^{\frac{3}{2}}} = \frac{y_{xx}}{\left(y_x^2 + 1\right)^{\frac{3}{2}}} := \kappa,$$

$$\iota y^{(3)} = \frac{\left(1 + y_x^2\right) y_{xxx} - 3\, y_x\, y_{xx}^2}{\left(1 + y_x^2\right)^3} := \kappa_s.$$

the Euclidean curvature κ and its derivative κ_s with respect to the arc-length $ds = \sqrt{y_x^2 + 1}$.

(κ, κ_s)-signature of a planar curve Γ is a planar curve $\mathcal{S}_\Gamma$ parametrized by

$$\left(\kappa|_\Gamma, \kappa_s|_\Gamma\right).$$

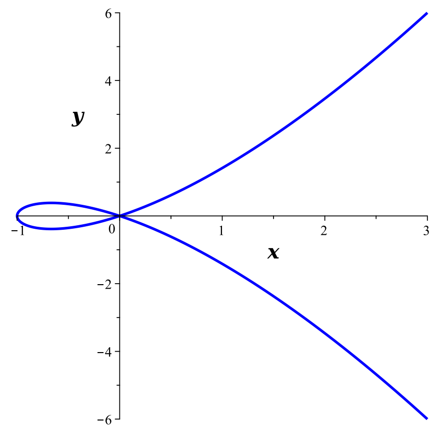
- For parametric curves $\Gamma = (x(t), y(t))$:

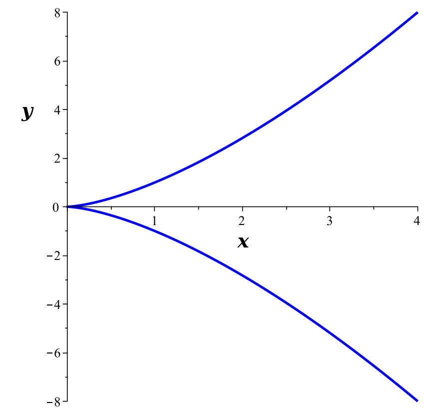
$$y_x|_\Gamma = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad y_{xx}|_\Gamma = \frac{\frac{dy_x}{dt}}{\frac{dx}{dt}}, \ldots .$$

- If Γ is defined implicitly by $F(x, y) = 0$:

$$y^{(1)}|_\Gamma = \frac{-F_x}{F_y}, \quad y^{(2)}|_\Gamma = \frac{-F_{xx}F_y^2 + 2F_{xy}F_xF_y - F_{yy}F_x^2}{F_y^3}, \ldots .$$

$$\Gamma_1: \quad x(t) = t^2 - 1,\ y(t) = t(t^2-1). \qquad \Gamma_2: \quad x(t) = t^2,\ y(t) = t^3.$$



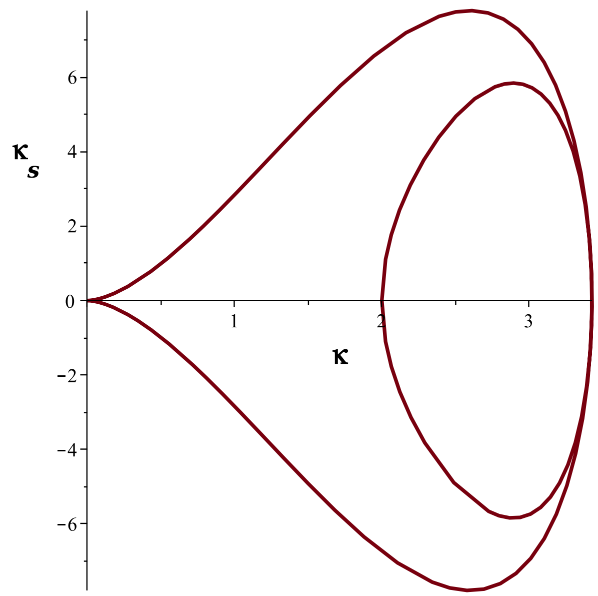


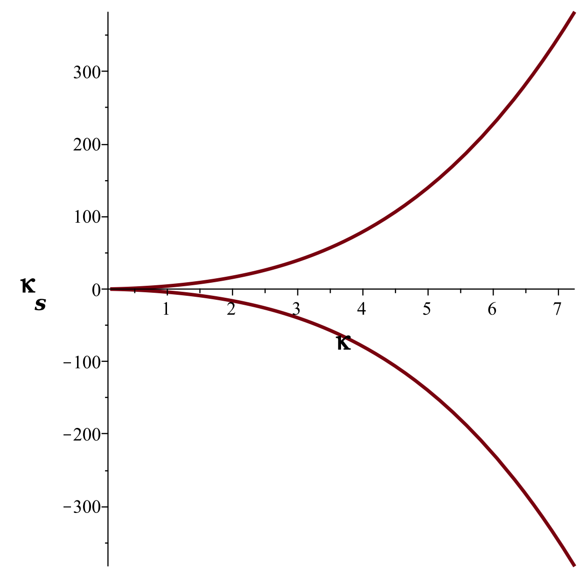
$$\mathcal{S}_{\Gamma_1}: \quad \kappa|_{\Gamma_1} = \frac{6t^2+2}{\sqrt{(9t^4-2t^2+1)^3}},$$

$$\kappa_s|_{\Gamma_1} = -24t\frac{9t^4+4t^2-1}{(9t^4-2t^2+1)^3}.$$

$$\mathcal{S}_{\Gamma_2}: \quad \kappa|_{\Gamma_2} = \frac{6}{\sqrt{t^2(9t^2+4)^3}},$$

$$\kappa_s|_{\Gamma_2} = -24\frac{9t^2+1}{(9t^2+4)^3t^3}.$$





Theorem:

For smooth or algebraic curves, congruent curves have equal signatures:

$$\Gamma_1 \underset{SE_2(\mathbb{R})}{\cong} \Gamma_2 \Longrightarrow \mathcal{S}_{\Gamma_1} = \mathcal{S}_{\Gamma_2}.$$

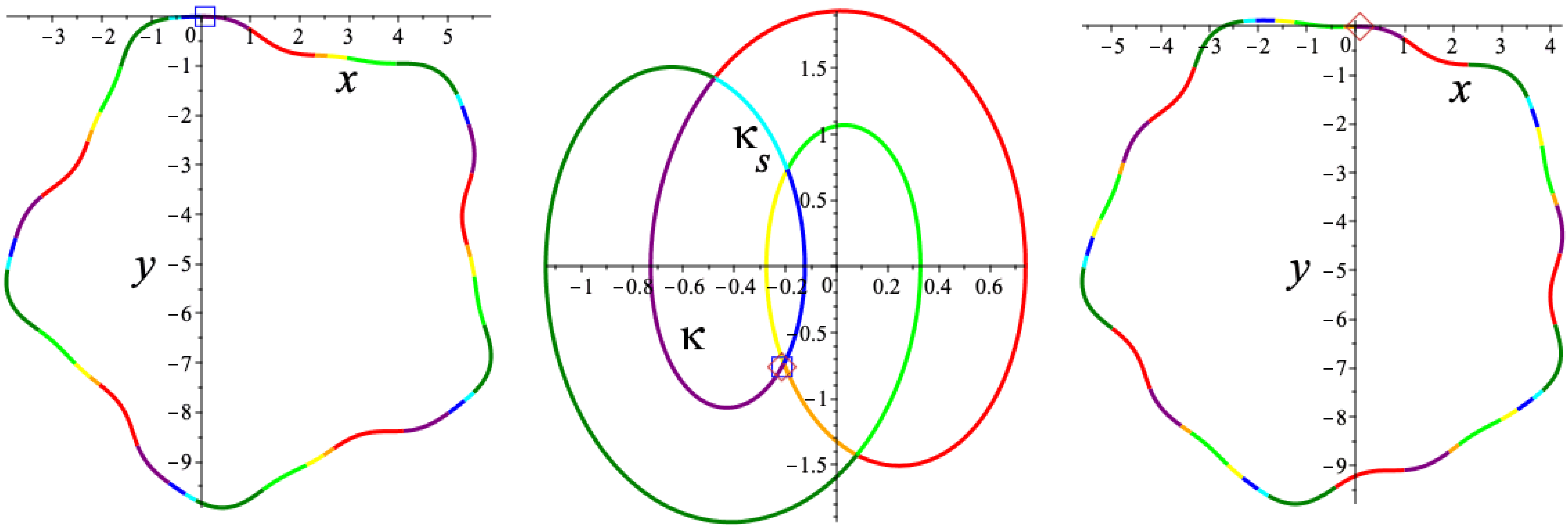
For analytic and algebraic curves, curves are congruent if and only if they have equal signatures:

$$\Gamma_1 \underset{SE_2(\mathbb{R})}{\cong} \Gamma_2 \Longleftrightarrow \mathcal{S}_{\Gamma_1} = S_{\Gamma_2}.$$

Challenge: decide whether or not two parametrized smooth curve are equal.

Non-congruent C^∞-smooth curves with identical (κ, κ_s)-signatures:

Γ_1 $\qquad$ $\mathcal{S}_{\Gamma_1} = \mathcal{S}_{\Gamma_2}$ $\qquad$ Γ_2



(Animation by Eric Geiger. The animation works well in Adobe Acrobat.)

E. Geiger and I. A. Kogan. “Non-congruent non-degenerate curves with identical signatures.” J. Math. Imaging Vision (2021)

See also: E. Musso and L. Nicolodi, “Invariant signatures of closed planar curves.” J. Math. Imaging Vision (2009)

Generalization to other groups ($\dim \mathcal{G} = r$):

Smooth: effective on open subsets smooth action $\alpha : \mathcal{G} \curvearrowright \mathbb{R}^2$ of a real Lie group.

- On J^r there exists two smooth local fundamental invariants for the prolonged action.

- Signatures based on these invariants provide a local solution to the equivalence problem for non-exceptional smooth planar curves and global solution for analytic curves.

E. Calabi, P. J. Olver, C. Shakiban, A. Tannenbaum, and S. Haker. ”Differential and numerically invariant signature curves applied to object recognition.” Int. J. Computer Vision (1998).

P. J. Olver. “The symmetry groupoid and weighted signature of a geometric object”. J. Lie Theory (2016).

Algebraic: effective rational action $\alpha : \mathcal{G} \curvearrowright \mathbb{C}^2$, $\mathcal{G} \subset PGL_2(\mathbb{C})$.

- On J^r there exists two algebraically independent rational invariants T_1 and T_2 such that $\mathbb{C}(J^r)^{\mathcal{G}} = \mathbb{C}(T_1, T_2)$.

- Signatures based on these invariants provide a global solution to the equivalence problem for non-exceptional algebraic planar curves.

- The implicit equation for the signature in principle (but often not in practice) can be computed by elimination.

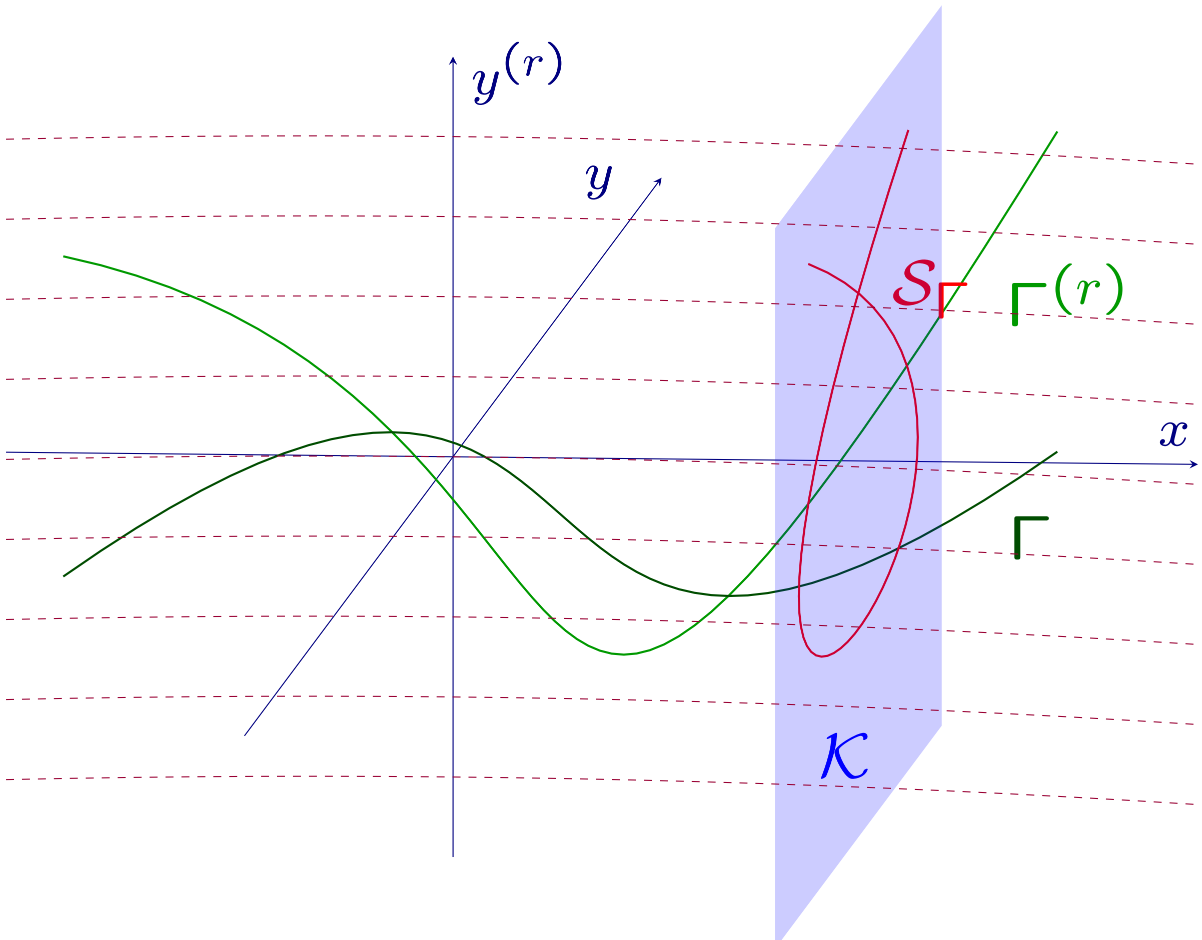
I. A. Kogan, M. Ruddy, and C. Vinzant. Differential signatures of algebraic curves. SIAM J. Appl. Algebra Geom. (2020)

T. Duff and M. Ruddy. “Signatures of algebraic curves via numeric algebraic geometry. J. Symbolic Comput. (2023).

Geometric interpretation of signatures based on normalized invariants:

i. Lift a curve to the r-th order jet space J^r.

ii. Project the lifted curve to the cross-section $\mathcal{K}$ along the orbits.



Symmetry detection:

The signature map $S\colon \Gamma \to \mathcal{S}_\Gamma$ can be used to determine the group of global symmetries of Γ, as well as the groupoid of its local symmetries.

- P. J. Olver. "The symmetry groupoid and weighted signature of a geometric object". J. Lie Theory (2016).

- I. A. Kogan, M. Ruddy, and C. Vinzant. Differential signatures of algebraic curves. SIAM J. Appl. Algebra Geom. (2020)

- E. Geiger and I. A. Kogan. "Non-congruent non-degenerate curves with identical signatures." J. Math. Imaging Vision (2021)

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- $\boxed{\text{Binary forms}}$

Signature solution for the GL_2-equivalence problem for binary forms:

Problem: Given two binary forms decide whether or not they can be matched to each other by a linear change of variables.

P. J. Olver. 1999. Classical Invariant Theory. Cambridge Univ. Press.

$$f_m(x,y) = \sum_{i=0}^{m} \binom{m}{i} c_i x^i y^{m-i} \quad \overset{dehomogenization}{\rightarrow} \quad \phi_m(p) = f_m(p,1)$$

The equivalence problem for binary forms of degree m under linear changes of variables

$$X = a_{11}x + a_{12}y, \quad Y = a_{21}x + a_{22}y$$

reduces to the equivalence problems for graphs of functions in p, q plane under:

$$P = \frac{a_{11}p + a_{12}}{a_{21}p + a_{22}}, \quad Q = \frac{q}{(a_{21}p + a_{22})^m}$$

GL_2-signature is based on a pair rational differential invariants (T_1, T_2) on J^4.

$$(T_1, T_2) = \left(\frac{T^2}{H^3}, \frac{V}{H^2} \right), \text{ where}$$

$$H = q\, q^{(2)} - \frac{m-1}{m} \left(q^{(1)} \right)^2$$

is, up to a constant multiple, the inhomogeneous counterpart of the Hessian of f_m, and

$$\begin{aligned} T &= q^2 q^{(3)} - 3\frac{m-2}{m} q q^{(1)} q^{(2)} + 2\frac{(m-1)(m-2)}{m^2} \left(q^{(1)} \right)^3, \\ V &= q^3 q^{(4)} - 4\frac{m-3}{m} q^2 q^{(1)} q^{(3)} + 6\, \frac{(m-2)(m-3)}{m^2} q \left(q^{(1)} \right)^2 q^{(2)} \\ &\quad - 3\, \frac{(m-1)(m-2)(m-3)}{m^3} \left(q^{(1)} \right)^4 . \end{aligned}$$

Theorem: Two binary forms can be mapped to each other by a linear change of variables if and only if their (T_1, T_2)-signature curves are equal.

Example 1

$$f(x,y) = x^4 + y^4 \quad \overset{dehomogenization}{\longrightarrow} \quad \phi(p) = p^4 + 1$$

$$T_1|_\phi = \frac{\left(p^4 - 1\right)^2}{3p^4}, \quad T_2|_\phi = \frac{p^8 - p^4 + 1}{6p^4}$$

$$\tilde{f}(x,y) = x^4 - y^4 \quad \overset{dehomogenization}{\longrightarrow} \quad \tilde{\phi}(p) = p^4 - 1$$

$$T_1|_{\tilde{\phi}} = -\frac{\left(p^4 + 1\right)^2}{3p^4}, \quad T_2|_{\tilde{\phi}} = -\frac{p^8 + p^4 + 1}{6p^4}$$

The implicit equation for both signatures is the same line:

$$6T_2 - 3T_1 - 1 = 0.$$

Over $\mathbb{R}$ the signatures of f and $\tilde{f}$ occupy two opposite rays of this line, while over $\mathbb{C}$ both signatures coincide with the entire line.

$$f \underset{GL_2(\mathbb{C})}{\cong} \tilde{f} \qquad \text{but} \qquad f \underset{GL_2(\mathbb{R})}{\ncong} \tilde{f}$$

Example 2

$$g(x,y) = x^4 + x^2y^2 + y^4 \quad \overset{dehomogenization}{\longrightarrow} \quad \psi(p) = p^4 + p^2 + 1$$

$$T_1|_\psi = \frac{324p^2\left(p^4-1\right)^2}{(2p^4+11p^2+2)^3}, \quad T_2|_\psi = \frac{3(16p^8+20p^6+9p^4+20p^2+16)}{2\left(2p^4+11p^2+2\right)^2}.$$

$$\tilde{g}(x,y) = x^4 - x^2y^2 + y^4 \quad \overset{dehomogenization}{\longrightarrow} \quad \tilde{\psi}(p) = p^4 - p^2 + 1$$

$$T_1|_{\tilde{\psi}} = -\frac{324p^2\left(p^4-1\right)^2}{(2p^4-11p^2+2)^3}, \quad T_2|_{\tilde{\psi}} = \frac{3(16p^8-20p^6+9p^4-20p^2+16)}{2\left(2p^4-11p^2+2\right)^2}.$$

The implicit equation for both signatures is the same cubic:

$$9800T_2^3 - 19773T_1^2 + 79092T_1T_2 - 64392T_2^2 - 13182T_1 + 33714T_2 - 972 = 0.$$

$$g \underset{GL_2(\mathbb{C})}{\cong} \tilde{g} \qquad \text{but} \qquad g \underset{GL_2(\mathbb{R})}{\not\cong} \tilde{g}$$

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