Classification of Curves in 3D via Affine Integral Invariant Signature

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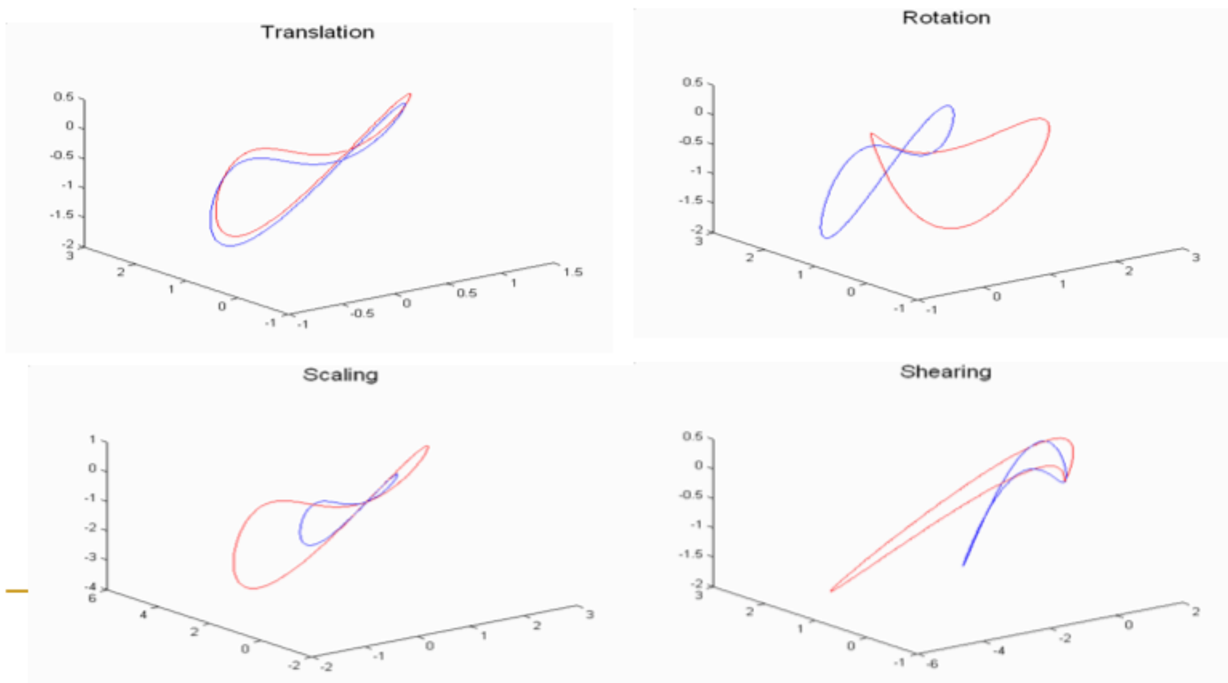
Abstract: We propose a robust classification algorithm for curves in 3D, under special and full affine groups of transformations. To each spatial curve we assign a plane signature curve. Curves, equivalent under an affine transformation, have the same signature. The signatures proposed here are based on integral invariants, which behave much better on noisy images than classically known differential invariants. Though the integral invariants for planar curves were known before [2], the affine integral invariants for spatial curves were computed by the authors for the first time [4],[5]. Using the inductive variation [3] of the moving frame method [1] we compute affine invariants in terms of Euclidean invariants. We present two types of signatures, the global signature and the local signature [5]. Both signatures are independent of parameterization (curve sampling). The global signature depends on the choice of the initial point and does not allow to compare fragments of the curves, and it is therefore sensitive to occlusions. The local signature, although slightly more sensitive to noise, is independent of the choice of the initial point and is not sensitive to the occlusion of the image. It allows to establish local equivalence of the curves.

Affine Transformations of Curves in $\mathbb{R}^3$

$$\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0.$$

include:



Previous Classification Methods

- Registration – finding a matching transformation

  – difficult

- Affine Signatures based on differential invariants – involve differentiation up to 5th order for volume preserving affine transformations – very sensitive to noise

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Proposed method:

Affine Signature based on integral invariants

not sensitive to noise

Integral Variables

We extend the affine transformations to integral variables up to the second order:

$$
\begin{aligned}
X^{(ijk)}(t) &= \int_a^t X(t)^i Y(t)^j Z(t)^k dX(t), \quad j+k \neq 0, i+j+k \leq 2 \\
Y^{(ijk)}(t) &= \int_a^t X(t)^i Y(t)^j Z(t)^k dY(t), \quad i+k \neq 0, i+j+k \leq 2 \\
Z^{(ijk)}(t) &= \int_a^t X(t)^i Y(t)^j Z(t)^k dZ(t), \quad i+j \neq 0, i+j+k \leq 2
\end{aligned}
$$

where

- integral is along a curve $\gamma(t) = (x(t), y(t), z(t))$, $t \in [a, b]$,
- $X = x(t) - x(a), Y = y(t) - y(a), Z = z(t) - z(a)$

Among 21 such variables we choose 11 independent: $Z^{(100)}, Z^{(010)}, Y^{(100)}, Z^{(011)}, Z^{(020)}, Z^{(101)}, Z^{(110)}, Y^{(101)}, X^{(110)}, X^{(101)}, X^{(020)}$.

Affine Integral Invariants for Curves in 3D

$$
\begin{aligned}
I_1 &= n_1 X + n_2 Z - n_3 Y \\
I_2 &= 2\,n_1(XYZ^2 - 3\,Z^{(011)}X + 3\,YZ^{(101)} - ZZ^{(110} - 2\,ZY^{(101)}) \\
&+ n_2(2\,XY^2Z + 3\,XZ^{(020)} - 6\,ZX^{(020} - 4\,YZ^{(110)} - 2\,YY^{(101}) \\
&- 2\,n_3(3\,YX^{(101)} - 3\,ZX^{(110} + XZ^{(110)} - XY^{(101)}) \\
I_3 &= \text{one page expression see [5].}
\end{aligned}
$$

where

$$
n_1 = \frac{YZ}{2} - Z^{(010)}, n_2 = \frac{XY}{2} - Y^{(100)}, n_3 = \frac{XZ}{2} - Z^{(100)}.
$$

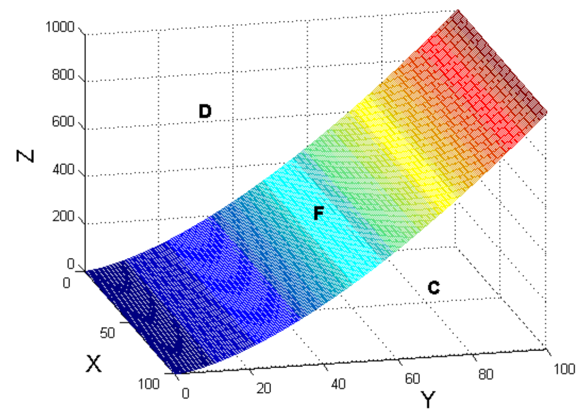
Remark: These are invariants with respect to the volume preserving (special) affine transformations. An appropriate normalization of these invariants by the range of I_1 is used to obtain invariants under the full affine group.

Geometric Interpretation of I_1

$\gamma(t) = (X(t), Y(t), Z(t)) \subset \mathbb{R}^3$ is a curve s.t. $\gamma(0) = (0, 0, 0)$.

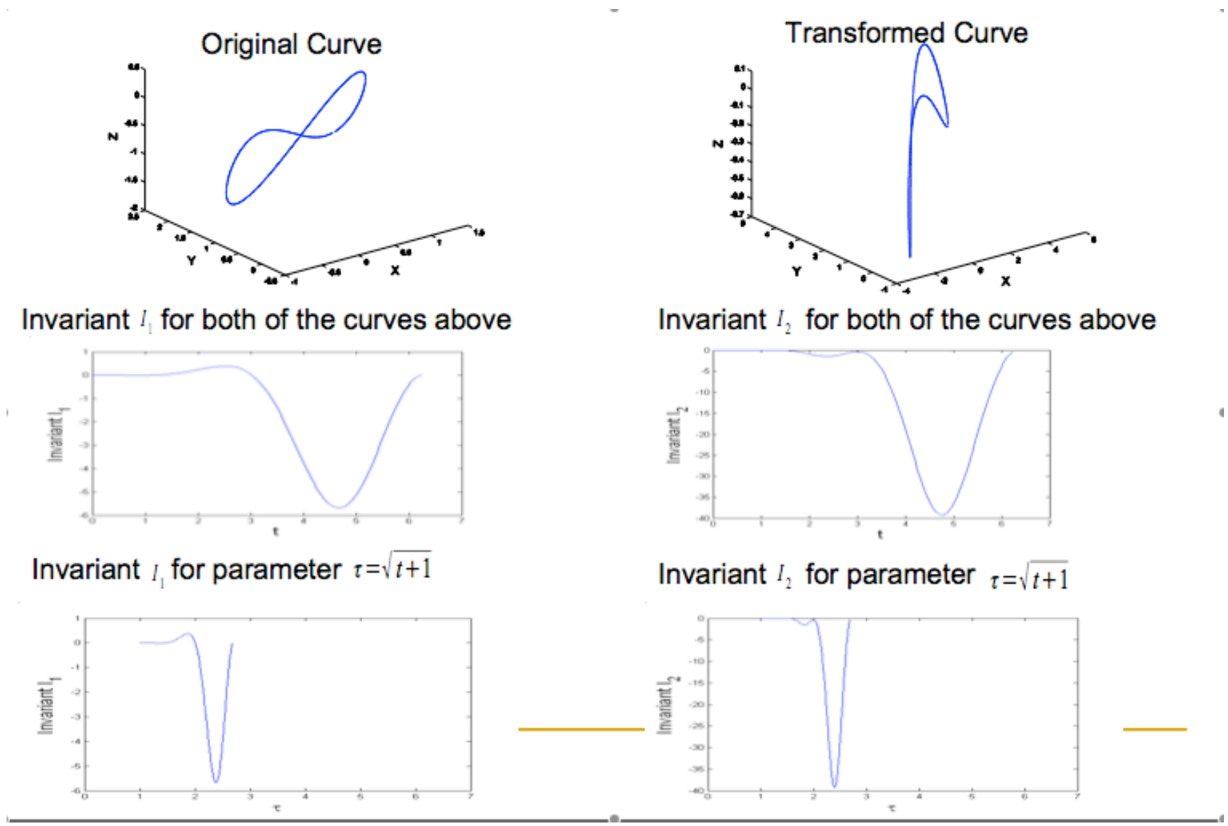
$I_1(t) = n_1(t)X(t) + n_2(t)Z(t) - n_3(t)Y(t)$

$n_1(t)\,X(t) = X\left(\frac{1}{2}YZ - \int_0^t Y\,dZ\right)$
is the (signed) volume C
"under" the surface $F = (Y(t), Z(t)) \times [0, X(t)]$.



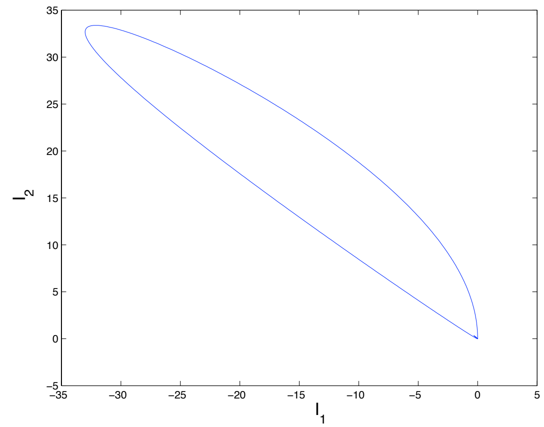
Two curves related by the affine transformation have the same invariants.

$$\gamma(t) = (\sin t - \frac{1}{5}\cos^2 t + \frac{1}{5}, \frac{1}{2}\sin t - \cos t + 1\ \sin^2 t + \cos t - 1)$$



Integral invariants depend on parameterization

Global Affine Signature – a plot of $I_1(t)$ vs $I_2(t)$:

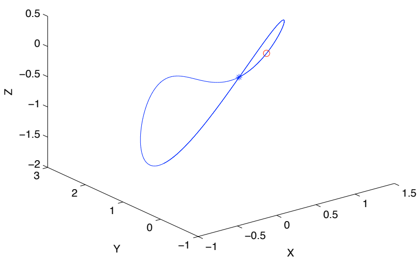


Independence of parametrization:

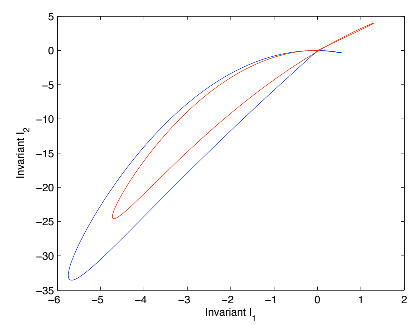
global signature for $\gamma(t)$, $\overline{\gamma(t)}$, $\gamma(\tau)$ and $\overline{\gamma(\tau)}$ coincide:

Dependence on the choice of the initial point:

Different choices of the initial point:



Different global signatures:



Localization of signature construction

- We define evaluation of an invariants $I_1^{[a,b]}$, $I_2^{[a,b]}$ and $I_3^{[a,b]}$ on a sub-segment of γ by treating the starting point of the segment as the initial point and computing the value of integral variables at the end point. Evaluation of an invariant on a segment is a real number.

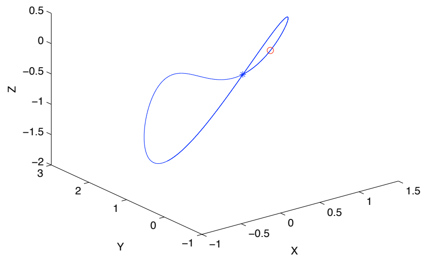
- We chose a small increment $\Delta > 0$ and use I_1 to partition γ into the equi-affine intervals such that evaluation $\left| I_1^{[a_j,a_{j+1}]} \right| = \Delta$, $i = 0, ..N$, where N is some integer.

- Local Affine Signature of γ is a discrete plane plot $(I_2(i), I_3(j))$, $j = 1..N$, where $I_2(j) = I_2^{[a_j,a_{j+1}]}$ and $I_3(j) = I_3^{[a_j,a_{j+1}]}$ are evaluations of invariants on the corresponding segments.

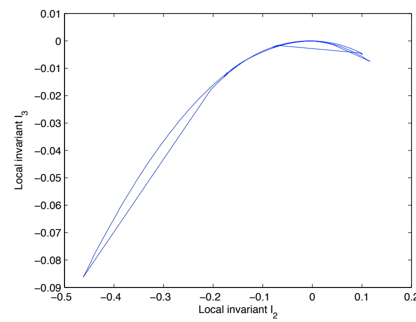
Local Integral Affine Signature

Independence of the initial point

Different choices of the initial point:



Same local signature:



Full Affine Invariants and Signatures

Independence of scaling and reflections

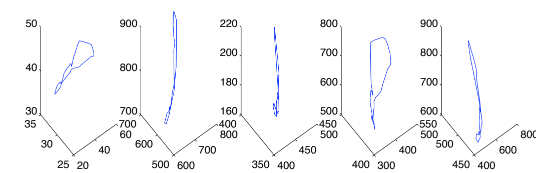
For $\lambda \in \mathbb{R}$ composition of scaling and reflection $(x, y, z) \longrightarrow (\lambda x, \lambda y, -\lambda z)$ induces transformation $I_1 \rightarrow -\lambda^3 I_1$, $I_2 \rightarrow \lambda^6 I_2$, $I_3 \rightarrow -\lambda^3 I_3$.

The following normalizations are invariant with respect to the full affine group

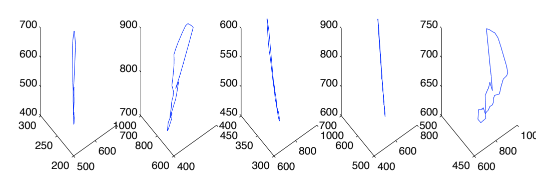
$$\hat{I}_1 = \frac{|I_1|}{max_t(|I_1|)}, \; \hat{I}_2 = \frac{I_2}{max_t(I_1^2)}, \; \hat{I}_3 = \frac{|I_3|}{max_t(|I_1|)}$$

Experimental Setting

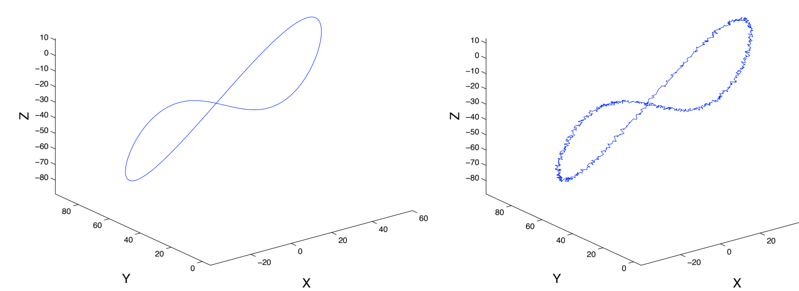
Extract 100 feature curves from 3D object in the Princeton Database as a training set:



Apply 9 randomly generated special affine transformations to each curve to generate the testing set:



Add zero mean Gaussian noise with standard deviation 0.1, 1, 2 to each testing curve:



Experimental Results:

Curve Classification in 3D under the Affine Transformations:

Classification error rate with same parametrization, same initial points

Noise variance	I_1	I_2	Global signature	Local signature
$\sigma = 0.5$	0.0022	0.0472	0.06	0.07
$\sigma = 1$	0.04	0.12	0.15	0.17
$\sigma = 2$	0.0789	0.2233	0.28	0.32

Classification error rate with different parametrization, same initial points:

Noise variance	I_1	I_2	Global signature	Local signature
$\sigma = 0.5$	0.42	0.61	0.06	0.07
$\sigma = 1$	0.48	0.70	0.15	0.17
$\sigma = 2$	0.56	0.83	0.28	0.32

Classification error rate with different parametrization, different initial points

Noise variance	I_1	I_2	Global signature	Local signature
$\sigma = 0.5$	0.87	0.95	0.95	0.07
$\sigma = 1$	0.91	0.97	0.97	0.17
$\sigma = 2$	0.94	0.98	0.98	0.32

Conclusion

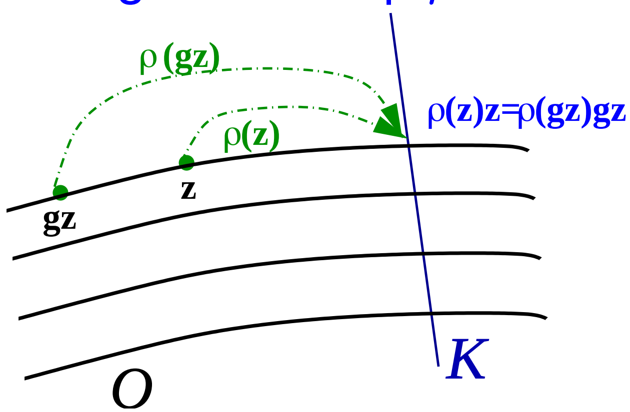
- Integral Affine Invariants provide a noise tolerant classification method for curves with respect to the affine transformations (for comparison: differential invariants give 80% error rate). *Integral invariants are functions of the parameter, and hence depend on the parameterization*.

- Global Integral Signature provides a classification method independent of parametrization (curve sampling). *Global Integral Signatures depend on the choice of the initial point.*

- Local Integral Signature provides a classification method independent of the choice of the initial point, they can be used on images with occlusions and for comparing fragments of the contours. They are slightly more sensitive to noise than global signatures.

Appendix: Computing Invariants

Moving frame map $\rho : U \rightarrow G$ defined by $\rho(z) \cdot z \in \mathcal{K}$



$G \curvearrowright M$ free $\Rightarrow \rho$ is smooth, G-equivariant:

$$\rho(g \cdot z) \cdot (g \cdot z) = \rho(z) \cdot z \overset{\text{freeness}}{\Longrightarrow} \rho(g \cdot z) = \rho(z)\, g^{-1}$$

Invariantization: $\iota f(z) = f(\rho(z) \cdot z)$

Normalized invariants: $\iota z_1, \ldots, \iota z_n$ contains a set of fundamental invariants.

Selected References:

1. M. Fels and P. J. Olver. Moving Coframes. II. Regularization and Theoretical Foundations. *Acta Appl. Math.*, 55: 127–208, 1999.

2. C. Hann and C. E. Hickerman. Projective curvature and integral invariants. *Acta applicandae mathematicae*, 74:177–193, 2002.

3. I. A. Kogan. Two algorithms for a moving frame construction. *Canad. J. Math.*, 55:266–291, 2003.

4. S. Feng, I. A. Kogan, and H. Krim. Integral invariants for 3D curves: an Inductive Construction. In *proceedings of IS&T/SPIE joint symposium*, San Jose, CA, 2007.

5. S. Feng, H. Krim. and I. A. Kogan Classification of curves in 2D and 3D via affine integral signatures. *in preparation.*