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Irina Kogan
North Carolina State University

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Based on:

1. J. M. Burdis, I. A. Kogan and H. Hong “Object-image correspondence for algebraic curves under projections“, *Symmetries, Integrability and Geometry: Methods and Applications (SIGMA)*, **9** (2013), 31pp

2. I. A. Kogan and P. J. Olver “Invariants of objects and their images under surjective maps“, *Lobachevskii Journal of Mathematics*, Vol 36, **3** (2015), 260–285.

available from `http://www.math.ncsu.edu/~iakogan`

Object-image correspondence problem for algebraic curves:

(aka "projection problem"):

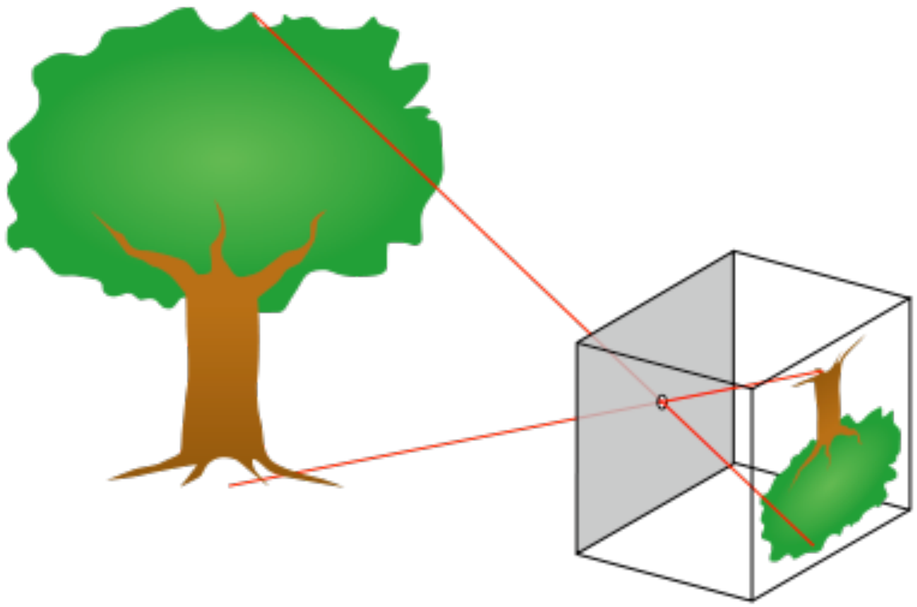
Given: An algebraic curve $\mathcal{Z} \subset \mathbb{R}^3$ and an algebraic curve $\mathcal{X} \subset \mathbb{R}^2$.

Decide: $\exists$? a central (or a parallel) projection $P \colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$\mathcal{X} = \overline{P(\mathcal{Z})}$$

- a central projection models a pinhole camera (or a finite camera);
- a parallel projection (or a weak perspective projections, or an affine camera) approximates finite camera when the distance between the object and camera is much larger than the depth of the object

Pinhole camera



11 degrees of freedom:

- location of the center (3 degrees of freedom);
- position of the image plane (3 degrees of freedom);
- choice of affine coordinates on the image plane up to overall scaling (5 degrees of freedom).

Camera model:

$$P\colon \mathbb{R}^3 \to \mathbb{R}^2 \qquad \begin{aligned} x &= \frac{p_{11}\,z_1 + p_{12}\,z_2 + p_{13}\,z_3 + p_{14}}{p_{31}\,z_1 + p_{32}\,z_2 + p_{33}\,z_3 + p_{34}}, \\ y &= \frac{p_{21}\,z_1 + p_{22}\,z_2 + p_{23}\,z_3 + p_{24}}{p_{31}\,z_1 + p_{32}\,z_2 + p_{33}\,z_3 + p_{34}}. \end{aligned} \tag{1}$$

12 parameters p_{ij}, equivalent up to scaling by a nonzero constant $p_{ij} \to \lambda p_{ij}$.

Projective camera model:

$$[P]\colon \mathbb{P}^3 \to \mathbb{P}^2 : \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix}$$

- [] – homogeneous coordinates.
- $\mathbb{R}^n \hookrightarrow \mathbb{P}^n$: $\mathbf{z} = (z_1, \ldots, z_n) \to [z_1, \ldots, z_n, 1] = [\mathbf{z}]$.
- points $[z_1, \ldots, z_n, 0] \in \mathbb{P}^n$ are said to be at infinity

Camera center:

$$[P] : \mathbb{P}^3 \to \mathbb{P}^2 : \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix}$$

- For P to be surjective, the 3×4 matrix must have rank 3.
- $[P] : \mathbb{P}^3 \to \mathbb{P}^2$ is undefined at the unique point $[z_1^0, z_2^0, z_3^0, z_4^0] \in \mathbb{P}^3$ – the center of the projection, such that $P\,(z_1^0, z_2^0, z_3^0, z_4^0)^T = (0,0,0)^T$.

Types of cameras: (Hartley R.I., Zisserman A., *Multiple View Geometry in Computer Vision*, 2004)

finite: center is not at $\infty \iff$ left 3×3 submatrix of P is non-singular (central projections with 11 degrees of freedom);

infinite: center is at ∞;

affine: center is at ∞ and the preimage of the line at ∞ in $\mathbb{P}^2$ is the plane at infinity in $\mathbb{P}^3 \iff$ the last row of $[P]$ is $[0,0,0,1]$, (parallel projections with 8 degrees of freedom).

How one can solve object-image correspondance problem for central projections (finite cameras)?

Straightforward approach: Set up a system of polynomial equations on 12 unknown projection parameters p_{ij}. Decide if the system has a real solution. (Eliminate 12 parameters.)

Our approach: Use the relation between the projection problem and the group-equivalence problem to set up the system of equations that involves the center of the projection only (3 parameters). Decide if the system has a real solution. (Eliminate 3 parameters.)

Advantage: We need to eliminate only 3 unknown parameters vs. 12 in the "straightforward" approach.

Projection criterion for algebraic curves

Given $\mathcal{Z} \subset \mathbb{R}^3$ and $\mathcal{X} \subset \mathbb{R}^2$, $\exists$ a central projection P such that $\mathcal{X} = \overline{P(\mathcal{Z})}$

$$\Updownarrow$$

$\exists\, c_1, c_2, c_3 \in \mathbb{R}$ such that $\mathcal{X}$ is $\mathcal{PGL}(3)$-equivalent to a planar curve:

$$\tilde{\mathcal{Z}}_c = \overline{\left(\frac{z_1 - c_1}{z_3 - c_3}, \quad \frac{z_2 - c_2}{z_3 - c_3}\right)}, \text{ where } (z_1, z_2, z_3) \in \mathcal{Z}$$

$\mathbf{c} = (c_1, c_2, c_3)$ is the center of the projection.

A is the left 3×3 submatrix of P,

$$P_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Note that $[A]$ belongs to $\mathcal{PGL}(3)$ and

$$[P_0][B][z_1, z_2, z_3,\ 1]^T = [z_1 - c_1, z_2 - c_2, z_3 - c_3]^T.$$

We reduced the $\boxed{\text{object-image correspondence problem}}$ to a

$\boxed{\text{group-equivalence problem with parameters}}$.

Group-equivalence problem for planar curves:

Let a group G act on $\mathbb{R}^2$. Given two planar algebraic curves $\mathcal{X}_1$, $\mathcal{X}_2$, decide if there exists $A \in G$ such that $\mathcal{X}_1 = \overline{A(\mathcal{X}_2)}$.

Proposed solution: is based on an algebraic adaptation of Cartan's equivalence method for solving local equivalence problem for smooth submanifolds.

Differential signatures of smooth curves in applications to computer vision:

[1] Calabi E., Olver P.J., Shakiban C., Tannenbaum A., Haker S., Differential and numerically invariant signature curves applied to object recognition (1998)

[2] Musso E., Nicolodi L., Invariant signature of closed planar curves (2009)

[3] Hoff D., Olver P.J., Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* (2013)

Algebraic signatures for algebraic curves in $\mathbb{R}^2$.

- Define a notion classifying set $\mathcal{I}$ of rational differential invariants.

- Define a notion exceptional curves with respect to the set $\mathcal{I}$.

- Use these invariants to define signatures of non-exceptional curves.

- Prove that $\boxed{\mathcal{X}_1 \cong_G \mathcal{X}_2 \quad \Longleftrightarrow \quad \mathcal{S}_{\mathcal{X}_1} = \mathcal{S}_{\mathcal{X}_2}}$ for non exceptional curves.

- Construct classifying sets of rational differential invariants for specific actions, e.g. $\mathcal{PGL}(3)$-action on $\mathbb{R}^2$:

$$\begin{aligned} \bar{x} &= \frac{a_{11}\, x + a_{12}\, y + a_{13}}{a_{31}\, x + a_{32}\, y + a_{33}}, \\ \bar{y} &= \frac{a_{21}\, x + a_{22}\, y + a_{23}}{a_{31}\, x + a_{32}\, y + a_{33}}. \end{aligned}$$

Jet variables:

Differential invariants depend on the derivatives: $y^{(k)}$ – k-the derivative of y with respect to x.

Let $F(x, y)$ be the implicit equation of $\mathcal{X}$ (not a vertical line). Then by implicit differentiation:

$$y_{\mathcal{X}}^{(1)} = -\frac{F_x}{F_y}, \qquad y_{\mathcal{X}}^{(2)} = \frac{-F_{xx}\,F_y^2 + 2\,F_{xy}\,F_x\,F_y - F_{yy}\,F_x^2}{F_y^3}, \qquad \ldots.$$

are rational functions on $\mathcal{X}$.

If $\mathcal{X}$ is a rational curve with parametrization $\big(x(t), y(t)\big)$, then

$$y^{(1)} = \frac{\dot{y}}{\dot{x}} \qquad , \ldots, \qquad y^{(k)} = \frac{\dot{y}^{(k-1)}}{\dot{x}},$$

are rational functions: $\mathbb{R} - \to \mathbb{R}$.

Classical G-curvatures and G-arc-lengths: *

$SE(2)$: $\kappa = \frac{y^{(2)}}{(1+[y^{(1)}]^2)^{3/2}}, \quad ds = \sqrt{1+[y^{(1)}]^2}\,dt \implies \kappa_s = \frac{d\kappa}{ds}, \kappa_{ss}, \ldots$

$\mathcal{SA}(2)$: $\mu = \frac{3\,\kappa\,(\kappa_{ss}+3\,\kappa^3)-5\,\kappa_s^2}{9\,\kappa^{8/3}}, \quad d\alpha = \kappa^{1/3} ds \implies \mu_\alpha = \frac{d\mu}{d\alpha}, \mu_{\alpha\alpha}, \ldots$

$\mathcal{PGL}(3)$: $\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha - 7\,\mu_{\alpha\alpha}^2 - 9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}, \quad d\rho = \mu_\alpha^{1/3} d\alpha \implies \eta_\rho = \frac{d\eta}{d\rho}, \ldots.$

Theorem:

- $\mathcal{I}_{\mathcal{PGL}} = \left\{K = \eta^3, \quad T = \eta_\rho\right\}$ is a classifying set of $\mathcal{PGL}(3)$ rational differential invariants.
- $\mathcal{I}_{\mathcal{PGL}}$-exceptional curves are lines and conics (parabola, an ellipse, or a hyperbola)

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*Inductive formulas I. K. Two algorithms for a moving frame construction, *Canad. J. of Math* (2003)

Classifying set of rational $\mathcal{PGL}(3)$-invariants:

$$\boxed{\Delta_2 = 9\,y^{(5)}\,[y^{(2)}]^2 - 45\,y^{(4)}\,y^{(3)}\,y^{(2)} + 40\,[y^{(3)}]^3.}$$

$$\begin{aligned}
K &= \frac{729}{8\,(\Delta_2)^8}\Big(18\,y^{(7)}\,[y^{(2)}]^4\,\Delta_2 - 189\,[y^{(6)}]^2\,[y^{(2)}]^6 \\
&+ 126\,y^{(6)}\,[y^{(2)}]^4\,(9\,y^{(5)}\,y^{(3)}\,y^{(2)} + 15\,[y^{(4)}]^2\,y^{(2)} - 25\,y^{(4)}\,[y^{(3)}]^2) \\
&- 189\,[y^{(5)}]^2\,[y^{(2)}]^4\,(4\,[y^{(3)}]^2 + 15\,y^{(2)}\,y^{(4)}) \\
&+ 210\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2\,(63\,[y^{(4)}]^2\,[y^{(2)}]^2 - 60\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} + 32\,[y^{(3)}]^4) \\
&- 525\,y^{(4)}y^{(2)}\,(9\,[y^{(4)}]^3\,[y^{(2)}]^3 + 15\,[y^{(4)}]^2\,[y^{(3)}]^2\,[y^{(2)}]^2 - 60\,y^{(4)}\,[y^{(3)}]^4\,y^{(2)} + 64\,[y^{(3)} \\
&+ 11200\,[y^{(3)}]^8\Big)^3;
\end{aligned}$$

$$\begin{aligned}
T &= \frac{243\,[y^{(2)}]^4}{2\,(\Delta_2)^4}\Big(2\,y^{(8)}\,y^{(2)}\,(\Delta_2)^2 \\
&- 8\,y^{(7)}\,\Delta_2\,(9\,y^{(6)}\,[y^{(2)}]^3 - 36\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2 - 45\,[y^{(4)}]^2\,[y^{(2)}]^2 + 120\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} - \\
&+ 504\,[y^{(6)}]^3\,[y^{(2)}]^5 - 504\,[y^{(6)}]^2\,[y^{(2)}]^3\,(9\,y^{(5)}\,y^{(3)}\,y^{(2)} + 15\,[y^{(4)}]^2\,y^{(2)} - 25\,y^{(4)}\,[y^{(3)}] \\
&+ 28\,y^{(6)}\,\big(432\,[y^{(5)}]^2\,[y^{(3)}]^2\,[y^{(2)}]^3 + 243\,[y^{(5)}]^2\,y^{(4)}\,[y^{(2)}]^4 - 1800\,y^{(5)}\,y^{(4)}\,[y^{(3)}]^3\,[y^{(2)} \\
&- 240y^{(5)}\,[y^{(3)}]^5\,y^{(2)} + 540y^{(5)}\,[y^{(4)}]^2\,[y^{(3)}]\,[y^{(2)}]^3 + 6600\,[y^{(4)}]^2\,[y^{(3)}]^4\,y^{(2)} - 2000y^{(4} \\
&- 5175\,[y^{(4)}]^3\,[y^{(3)}]^2\,[y^{(2)}]^2 + 1350\,[y^{(4)}]^4\,[y^{(2)}]^3\big) - 2835\,[y^{(5)}]^4\,[y^{(2)}]^4 \\
&+ 252\,[y^{(5)}]^3y^{(3)}\,[y^{(2)}]^2\,(9y^{(4)}\,y^{(2)} - 136\,[y^{(3)}]^2) - 35840\,[y^{(5)}]^2\,[y^{(3)}]^6 \\
&- 630\,[y^{(5)}]^2\,[y^{(4)}]\,[y^{(2)}]\,(69\,[y^{(4)}]^2\,[y^{(2)}]^2 - 160\,[y^{(3)}]^4 - 153\,y^{(4)}\,[y^{(3)}]^2\,[y^{(2)}]) \\
&+ 2100\,y^{(5)}\,[y^{(4)}]^2\,y^{(3)}\,(72\,[y^{(3)}]^4 + 63\,[y^{(4)}]^2\,[y^{(2)}]^2 - 193\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)}) \\
&- 7875\,[y^{(4)}]^4\,(8\,[y^{(4)}]^2\,[y^{(2)}]^2 - 22y^{(4)}\,[y^{(3)}]^2\,[y^{(2)}] + 9\,[y^{(3)}]^4)\Big).
\end{aligned}$$

Projective signature of planar curves.

- Invariants $K|_{\mathcal{X}}$ and $T|_{\mathcal{X}}$ are rational functions on $\mathcal{X}$ unless

  $\Delta_2|_{\mathcal{X}} = 0 \iff \mathcal{X}$ is a line or a conic (exceptional curves)).

- $\mathcal{PGL}(3)$-signature of a non-exceptional curve $\mathcal{X}$ is the image $\mathcal{S}_{\mathcal{X}}$ of the rational map

$$S|_{\mathcal{X}} = (K|_{\mathcal{X}}\,,\, T|_{\mathcal{X}}) \colon \mathcal{X} \to \mathbb{R}^2.$$

Theorem. If $\mathcal{X}_1$ and $\mathcal{X}_2$ have degree greater than 2 then

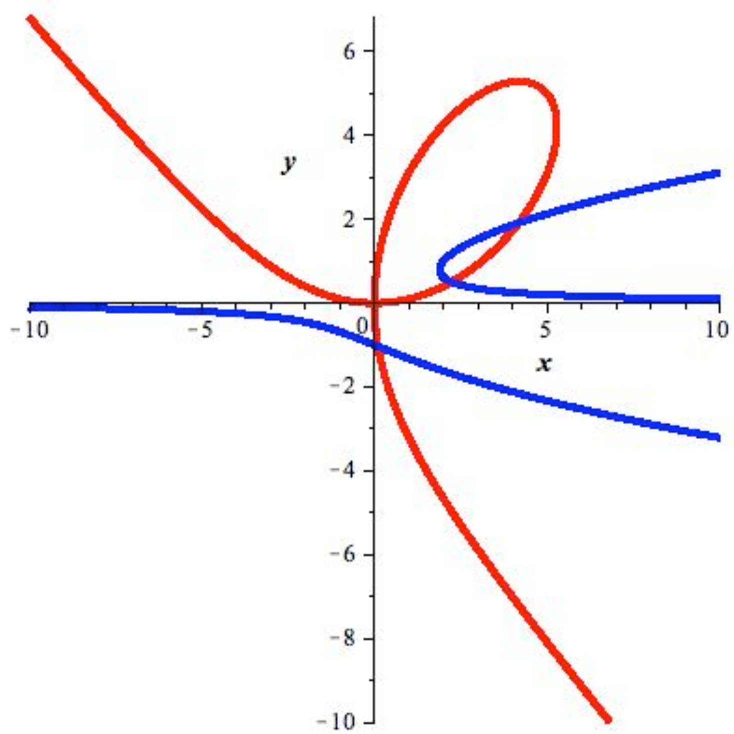
$$\boxed{\mathcal{X}_1 \underset{\mathcal{PGL}(3)}{\cong} \mathcal{X}_2 \quad\iff\quad \mathcal{S}_{\mathcal{X}_1} = \mathcal{S}_{\mathcal{X}_2}.}$$

Examples of solving $\mathcal{PGL}(3)$-equivalence problem

Is $\alpha(t) = \left(\frac{10\,t}{t^3+1}, \frac{10\,t^2}{t^3+1}\right)$ implicitly defined by $x^3 + y^3 - 10\,x\,y = 0$

$\mathcal{PGL}(3)$-equivalent to

$\beta(s) = \left(\frac{s^3+3\,s^2+3\,s+2}{s+1}, \; s+1\right)$ implicitly defined by $y^3 - x\,y + 1 = 0$?



- The signature $\mathcal{S}_\alpha$ for $\alpha(t) = \left(\frac{10\,t}{t^3+1}, \frac{10\,t^2}{t^3+1}\right)$ is a parametric curve

$$\begin{aligned} K|_\alpha(t) &= -\frac{9261}{50}\,\frac{(t^6-t^3+1)^3\,t^3}{(t^3-1)^8}, \\ T|_\alpha(t) &= -\frac{21}{10}\,\frac{(t^3+1)^4}{(t^3-1)^4}. \end{aligned}$$

- The signature $\mathcal{S}_\beta$ for $\beta(s) = \left(\frac{s^3+3\,s^2+3\,s+2}{s+1}, s+1\right)$ is a parametric curve

$$\begin{aligned} K|_\beta(s) &= -\frac{9261}{50}\,\frac{1}{(s^2+3\,s+3)^8\,s^8} \\ &\quad (s^9+9\,s^8+36\,s^7+83\,s^6+120\,s^5+111\,s^4+65\,s^3+24\,s^2+6\,s+1), \\ &\quad (s^6+6\,s^5+15\,s^4+19\,s^3+12\,s^2+3\,s+1)^2 \\ T|_\beta(s) &= -\frac{21}{10}\,\frac{(s^3+3\,s^2+3\,s+2)^4}{(s^2+3\,s+3)^4\,s^4}. \end{aligned}$$

- Is it true that $\mathcal{S}_\alpha = \mathcal{S}_\beta$ and hence α and β are $\mathcal{PGL}(3)$-equivalent?
  - $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ have the same implicit equation:

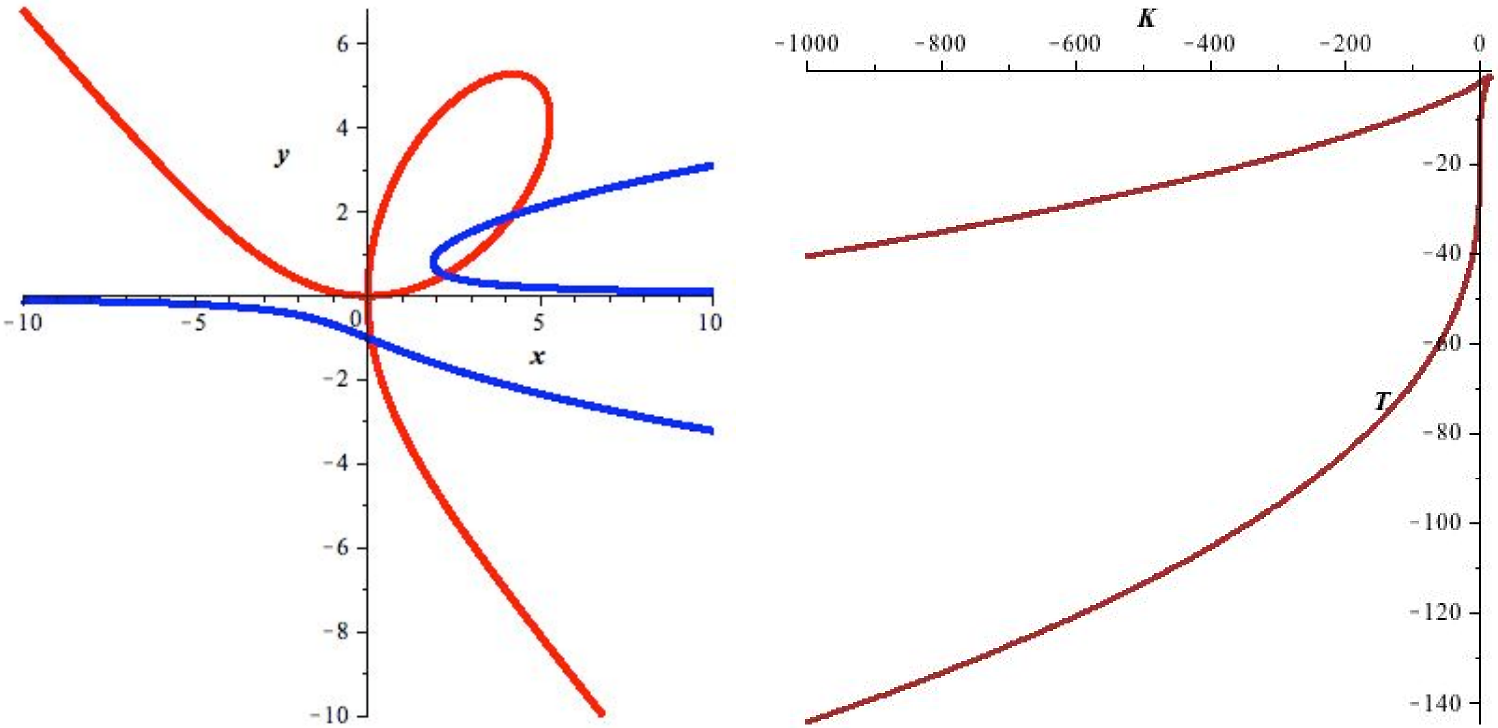
$$\begin{aligned} 0 &= 62523502209 + 39697461720\,T - 6401203200\,K + 5250987000\,T^2 \\ &- 2032128000\,K\,T + 163840000\,K^2 + 259308000\,T^3 + 53760000\,K\,T^2 \\ &+ 4410000\,T^4 \end{aligned}$$

Over $\mathbb{C}$ it is a sufficient condition, but not over $\mathbb{R}$.

We can look for a real reparameterization $t = \phi(s)$ by solving $K|_\alpha(t) = K|_\beta(s)$ and $T|_\alpha(t) = T|_\beta(s)$ for t in terms of s: $t = s + 1$ indeed works. Yes!!!

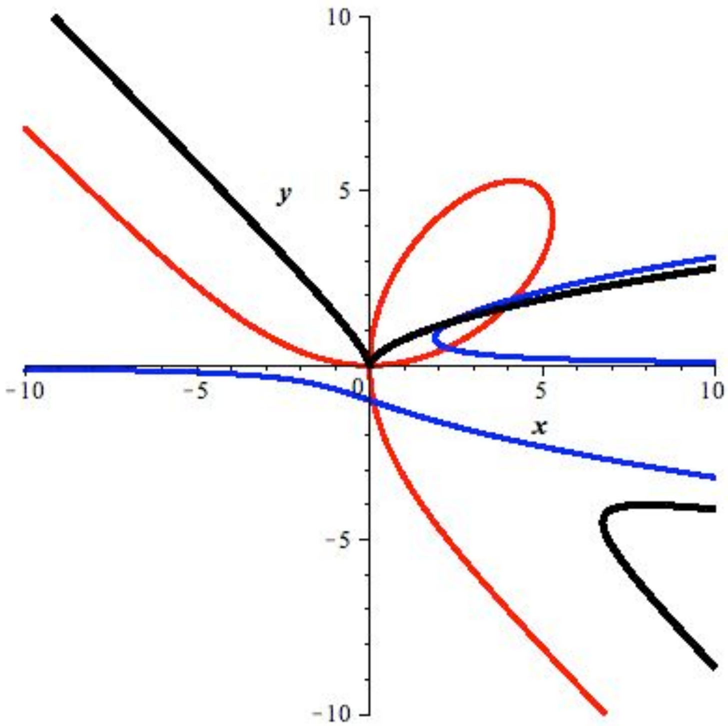
The $\mathcal{PGL}(3)$ transformation that brings α to β is

$$x \to \frac{10\,y}{x}, \quad y \to \frac{10}{x}.$$



Curves $\mathcal{X}_1$ and $\mathcal{X}_2$ have the same signature

Is $\gamma(w) = \left(\frac{w^3}{w+1}, \frac{w^2}{w+1}\right)$ implicitly defined by $y^3 - x^2 + x\,y^2 = 0$ $\mathcal{PGL}(3)$-equivalent to α and β?



No! because its signature is different:

$$K|_\gamma(w) = \frac{250047}{12800} \text{ and } T|_\gamma(w) = 0$$

and so $\mathcal{S}_\gamma = \left(\frac{250047}{12800}, 0\right)$ is a point!

Returning to the projection problem ...

Algorithm for central projections (rational curves).

INPUT: Rational parameterizations $\big(z_1(s),\, z_2(s),\, z_3(s)\big) \in \mathbb{Q}(s)^3$ and $\big(x(t),\, y(t)\big) \in \mathbb{Q}(t)^2$ of algebraic curves $\mathcal{Z} \subset \mathbb{R}^3$ and $\mathcal{X} \subset \mathbb{R}^2$, where $\mathcal{Z}$ is not a line.

OUTPUT: The truth of the statement:

$$\exists \text{ central projection } P \text{ such that } \mathcal{X} = \overline{P(\mathcal{Z})}.$$

NON-RIGOROUS OUTLINE:

1. if $\mathcal{X}$ is a line then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\mathcal{Z}$ is coplanar.
2. $\epsilon_c := \left(\frac{z_1(s)-c_1}{z_3(s)-c_3}, \frac{z_2(s)-c_2}{z_3(s)-c_3}\right)$ is a family of parametric curves.
3. if $\mathcal{X}$ is a conic then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c = (c_1, c_2, c_3)$, such that $\epsilon_c(s)$ parametrizes a conic.
4. if $\mathcal{X}$ is neither a line or a conic then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c$ such that the signature of the curve parametrized by $\epsilon_c(s)$ equals to the signature of $\mathcal{X}$.

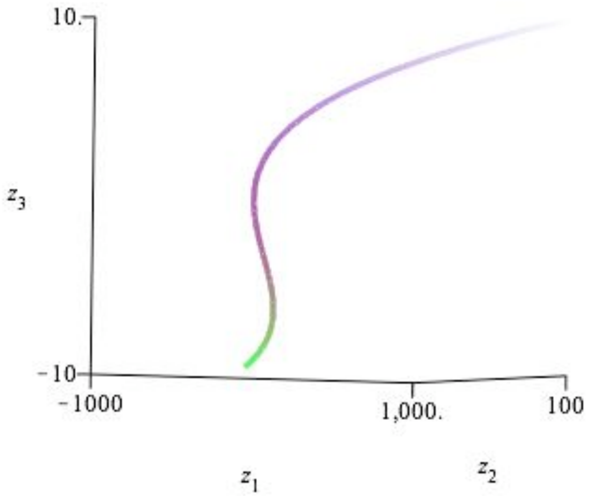
Maple Code over $\mathbb{C}$ is available at

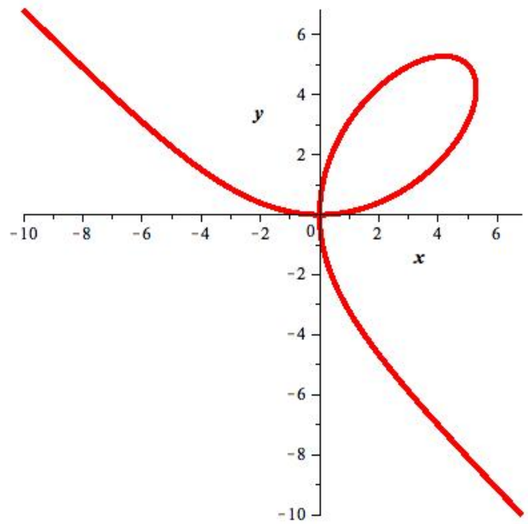
`http://www.math.ncsu.edu/~iakogan/symbolic/projections.html`

Example: central projections of the twisted cubic

Can the twisted cubic $\mathcal{Z}$ parametrized by

$$\Gamma(s) = \left(s^3,\, s^2,\, s\right) \,,\, s \in \mathbb{R}$$





be projected to a curve $\mathcal{X}_1$ parametrized by $\alpha(t) = \left(\frac{10\,t}{t^3+1}, \frac{10\,t^2}{t^3+1}\right)$ with an implicit equation $x^3 + y^3 - 10\,y\,x = 0$?

- Define the family of curves
  $\epsilon_c(s) = \left(\frac{z_1(s)-c_1}{z_3(s)-c_3}, \frac{z_2(s)-c_2}{z_3(s)-c_3}\right) = \left(\frac{s^3-c_1}{s-c_3}, \frac{s^2-c_2}{s-c_3}\right)$
- Compute invariants $K|_\epsilon(c,s)$ and $T|_\epsilon(c,s)$ with indeterminant values of c.
- The signature of $\mathcal{X}_1$ is parametrized by invariants:

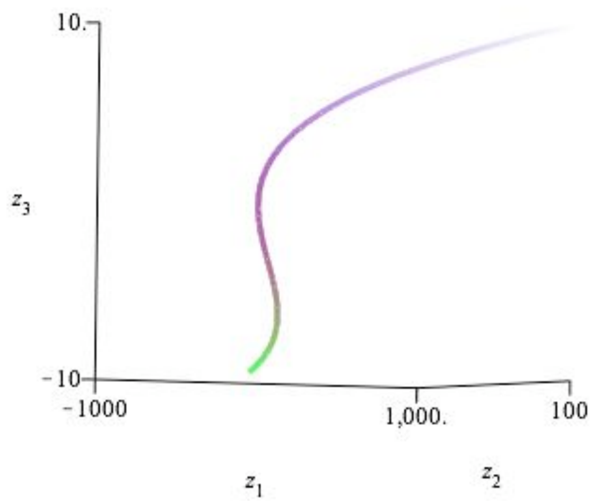
$$K|_\alpha(t) = -\frac{9261}{50}\frac{(t^6-t^3+1)^3\,t^3}{(t^3-1)^8},\; T|_\alpha(t) = -\frac{21}{10}\frac{(t^3+1)^4}{(t^3-1)^4}.$$

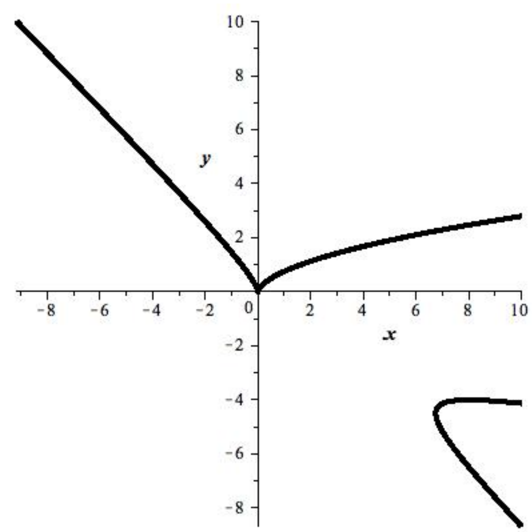
- $\exists ?\, c$ such that $(K|_\epsilon(c,s),\, T|_\epsilon(c,s))$ parametrize the same signature as $(K|_\alpha(t),\, T|_\alpha(t))$?
- This is indeed true for c=(-1,0,0).
- Yes!! The twisted cubic can be projected to $x^3 + y^3 - 10\,y\,x = 0$.
- A possible projection is $x = \frac{10\,z_3}{z_1+1}$, $y = \frac{10\,z_2}{z_1+1}$.

It follows that the twisted cubic can be projected to $\mathcal{X}_2$ because $\mathcal{X}_1 \underset{\mathcal{PGL}(3)}{\cong} \mathcal{X}_2$.

Can the twisted cubic $\mathcal{Z}$ parametrized by

$$\Gamma(s) = \left(s^3, s^2, s\right), s \in \mathbb{R}$$





be projected to a curve $\mathcal{X}_3$ parametrized by $\gamma(t) = \left(\frac{t^3}{t+1}, \frac{t^2}{t+1}\right)$ with an implicit equation $y^3 + y^2 x - x^2 = 0$?

- The signature of $\mathcal{X}_3$ degenerates to a point.

$$K|_\gamma(t) = \frac{250047}{12800} \text{ and } T|_\gamma(t) = 0, \quad \forall t \in \mathbb{R}.$$

- $\exists?\, c$ such that a curve parametrized by $\epsilon_c(s) = \left(\frac{s^3 - c_1}{s - c_3}, \frac{s^2 - c_2}{s - c_3}\right)$ has the same constant invariants as $\mathcal{X}_3$?

- The signature of $\mathcal{X}_3$ degenerates to a point.

$$K|_{\gamma}(t) = \frac{250047}{12800} \text{ and } T|_{\gamma}(t) = 0, \quad \forall t \in \mathbb{R}.$$

- $\exists ?\ c$ such that a curve parametrized by $\epsilon_c(s) = \left(\frac{s^3 - c_1}{s - c_3}, \frac{s^2 - c_2}{s - c_3}\right)$ has the same constant invariants as $\mathcal{X}_3$?

- This is indeed true for c=(0,0,-1).

- Yes!! The twisted cubic can be projected to $y^3 + y^2 x - x^2 = 0$.

- A possible projection is $x = \frac{z_1}{z_3+1}$, $y = \frac{z_2}{z_3+1}$.

- Recall that $\mathcal{X}_3$ is <u>not</u> $\mathcal{PGL}(3)$-equivalent to $\mathcal{X}_1$ and $\mathcal{X}_2$.

Can the twisted cubic be projected to a parabola parametrized by (t, t^2)?

- Does there exists c such that a curve parametrized by

$$\epsilon_c(s) = \left(\frac{s^3 - c_1}{s - c_3}, \frac{s^2 - c_2}{s - c_3} \right)$$

  is a quadric?

- Yes!! $\epsilon_c(s)$ is a conic iff $c_1 = a^3, \quad c_2 = a^2, \quad c_3 = a$ for all $a \in \mathbb{R}$.

- A projection of a twisted cubic is a conic if and only if the center of the projections is located on the twisted cubic!

Can the twisted cubic be projected to quintic parameterized by (t, t^5)?

- The signature of the quintic degenerates to a point:

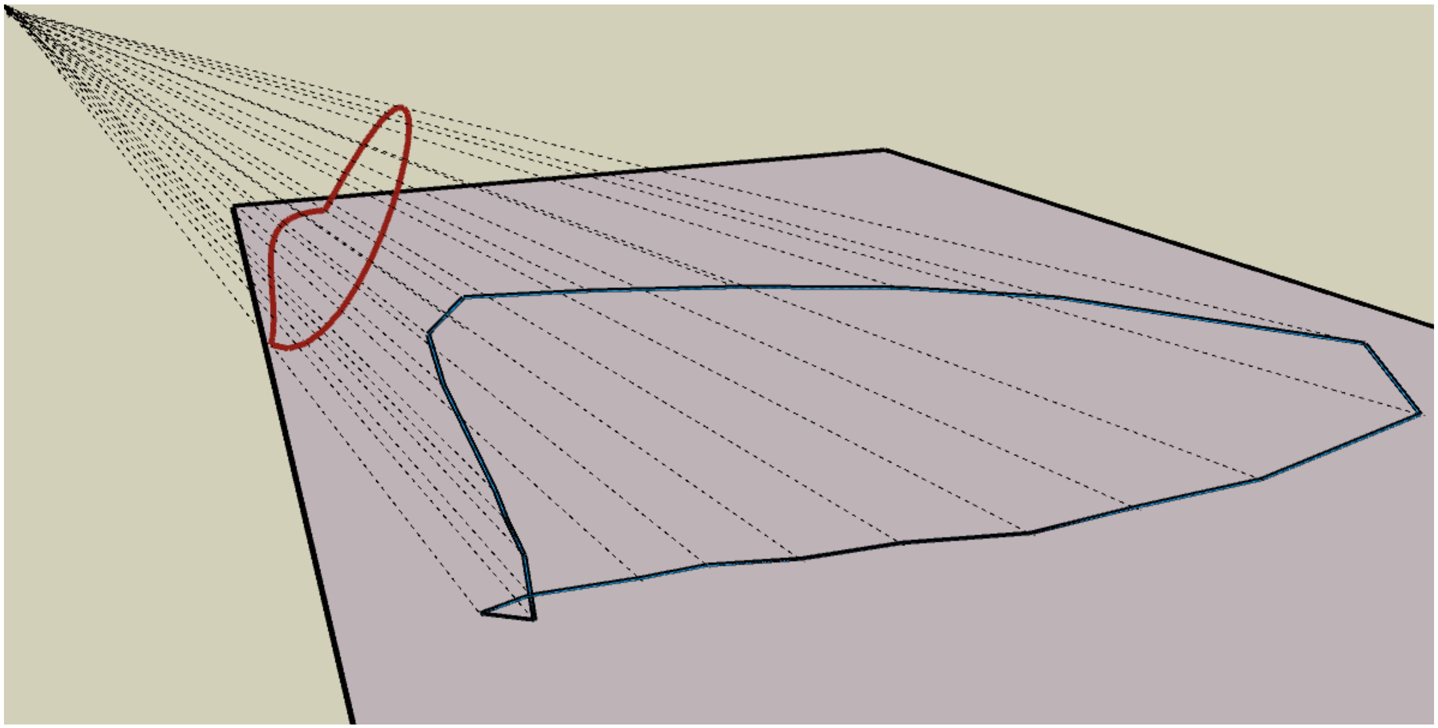
$$K(t) = \frac{1029}{128} \text{ and } T(t) = 0\,, \quad \forall t.$$

- Does there exists c such that

$$K|_\epsilon(c,s) = \frac{1029}{128} \text{ and } T|_\epsilon(c,s) = 0, \; \forall s \in \mathbb{R}?$$

- NO!! Substitution of several values of s gives an inconsistent system on c.

Relations between invariants of object and image



Invariants with respect to which group-action on $\mathbb{R}^3$? on $\mathbb{R}^2$?

- on $\mathbb{R}^3$ - standard linear action of $\mathcal{GL}(3)$(centro-affine invariants) or $\mathcal{SL}(3)$-action (centro-equi-affine invariants)
- on $\mathbb{R}^2$ - projective action (projective invariants)

Centro-equi-affine invariants for space curves in terms of the invariants of the planar images:

Theorem: Differential algebra of centro-equi-affine invariants of space is generated by:

- $\widehat{\eta} = P_0^*(\eta)$
- $\zeta = z_3\, P_0^* \left(\frac{1}{\mu_\alpha^{1/3}} \right)$
- $d\widehat{\rho} = P_0^*(d\rho)$,

where

- η and $d\rho$ are planar projective curvature and arc-length;
- μ and $d\alpha$ are planar equi-affine curvature and arc-length;
- P_0 is the standard central projection $x = \frac{z_1}{z_3}$, $x = \frac{z_2}{z_3}$ from the origin to the plane $z_3 = 1$:

Centro-equi-affine curvature, torsion and arc-lengths: *

Let $\mathcal{Z} \subset \mathbb{R}^3$ be parametric curve $\mathbf{z}(t) = ((z_1(t), z_2(t), z_3(t))$, then

- centro-equi-affine arc-lengths $dS := |\mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}|\, dt$ (undefined when $\mathcal{Z}$ is contained in the plane spanned by $\mathbf{z}(0)$ and $\dot{\mathbf{z}}(0)$).

- centro-equi-affine torsion $\tau = |\mathbf{z}_S, \mathbf{z}_{SS}, \mathbf{z}_{SS}|$ ($\tau \equiv 0 \iff \mathcal{Z}$ is coplanar).

- centro-equi-affine curvature $\kappa = |\mathbf{z}, \mathbf{z}_{SS}, \mathbf{z}_{SS}|$

Theorem κ, τ and dS generate differential algebra of centro-affine invariants.

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*Olver, P. J. Moving frames and differential invariants in centro-equi-affine geometry, *Lobachevskii J. of Math.* (2010)

Relationship between two generating sets:

- $\widehat{\eta} = \dfrac{a_{ss}\, a - \frac{7}{6}\, a_s^2 - \frac{3}{2}\, \kappa\, a^2}{3^{2/3}\, a^{8/3}}$;

- $\zeta = (3a)^{-1/3}$;

- $d\widehat{\rho} = (3\, a)^{1/3}\, dS$;

where $a = \kappa_S + 2\,\tau$ is identically zero iff $P_0(\mathcal{Z})$ is a line or a conic.

Thank you!