Affine Integral Invariants for Curves in 3D

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Outline

1. Definition and computation of integral invariants (inductive approach)

2. Application to curve classification (signatures)

Affine Action on Curves in $\mathbb{R}^n$

$A(n) = GL(n) \ltimes \mathbb{R}^n$ acts on $\mathbb{R}^n$:

For $g = (M, \mathbf{v})$: $\mathbf{M} \in \mathbf{GL(n)}$, $\mathbf{v} \in \mathbb{R}^{\mathbf{n}}$ and $p \in \mathbb{R}^n$:
$g \cdot \mathbf{x} = M\mathbf{x} + \mathbf{v} = \overline{\mathbf{x}}$.

E.g. On $\mathbb{R}^2$: $g = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$, s.t $a_{11}a_{22} - a_{21}a_{12} \neq 0$

$(x_1, x_2) \mapsto (a_{11}x_1 + a_{12}x_2 + v_1,\ a_{21}x_1 + a_{22}x_2 + v_2)$ Induced action on curves in $\mathbb{R}^n$
$\gamma(t) = (x_1(t), \ldots, x_n(t)) \rightarrow g \cdot \gamma(t) = g \cdot (x_1(t), \ldots, x_n(t))$.

Equivalence and Symmetry

γ and $\bar{\gamma}$ affine equivalent $\Leftrightarrow \exists g \in A(n)$ s.t. $\bar{\gamma} = g \cdot \gamma$

$g \in A(n)$ is a symmetry of γ if $\gamma = g \cdot \gamma$.

More generally:

G -a Lie group acting on a manifold M.

$G \curvearrowright M \Rightarrow G \curvearrowright \{p$-dimensional submanifolds of $M\}$.

Equivalence: $N \sim \bar{N} \iff \exists g \in G : \bar{N} = g \cdot N$

Symmetry of N: $G_N = \{h \in G |\, h \cdot N = N\}$

$\bar{N} = g \cdot N \iff G_{\bar{N}} = g G_N g^{-1}$.

Key to the solution: Invariants.

Prolongation of the action

$G \curvearrowright \mathbb{R}^n$: $\bar{\mathbf{x}} = g \cdot \mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$

To jet spaces $\Rightarrow$ differential invariants (e.g Euclidean curvature)

$G \curvearrowright \mathbb{R}^n \Rightarrow G \curvearrowright J^l(\mathbb{R}^n, 1) \cong \mathbb{R}^N$, where $N = n(l+1)$

parametrized by $(x_i^{(j)}, i = 1..n, j = 0, .., l)$ where $x_i^{(0)} = x_i$

A curve $\mathbf{x}(t) \subset \mathbb{R}^n$ is lifted to $J^l(\mathbb{R}^n, 1)$ by $x_i^{(j+1)}(t) = \frac{d{x_i}^{(j)}(t)}{dt}$

The action is prolonged by $\overline{x_i^{(j+1)}}(t) = \frac{d\overline{{x_i}^{(j)}}(t)}{dt}$

To product space $\Rightarrow$ joint invariants (e.g Euclidean distance)

$G \curvearrowright \mathbb{R}^n \Rightarrow G \curvearrowright \mathbb{R}^n \times \cdots \times \mathbb{R}^n : g(\mathbf{x}_1, \ldots, \mathbf{x}_l) = (g \cdot \mathbf{x}_1, \ldots g \cdot \mathbf{x}_l)$

To potential space $\Rightarrow$ integral invariants *Hann and Hickerman (2002)*

$G \curvearrowright \mathbb{R}^n \Rightarrow G \curvearrowright P^l(\mathbb{R}^n, 1) \subset \mathbb{R}^N$, *where* $N = n\left[\frac{(n+l)!}{l!n!} - l + 1\right]$

$\mathbb{R}^N$ *is parametrized by*
$\big(x_1^0, \ldots, x_n^0,\, x_1, \ldots, x_n, X_i^K | i = 1..n, K = (k_1, \ldots, k_n) \in \mathbb{N}^n,$
s.t $|K| = k_1 + \cdots + k_n \leq l$ *and* $|K| - k_i > 0\big)$

A curve $\mathbf{x}(t) \subset \mathbb{R}^n$ *is lifted to* $P^l(\mathbb{R}^n, 1)$ *by* $x_i^0(t) = x_i(0)$ *and*

$$X_i^K(t) = \int_0^t \mathbf{x}^K(t) dx_i(t), \quad \mathbf{x}^K = x_1^{k_1} \cdots x_n^{k_n}.$$

The action on $\mathbb{R}^n$ *is prolonged by:*

$$\overline{\mathbf{x}} = g \cdot \mathbf{x}, \quad \overline{\mathbf{x}}^0 = g \cdot \mathbf{x}^0.$$

$$\overline{X}_i^K(t) = \int_0^t \overline{\mathbf{x}}(t)^K \overline{dx_i}(t)$$

Hann and Hickerman (2002) group action axioms are satisfied. Relations on variables that correspond to the integration by parts are respected. Computed invariants when $n = 2$.

Example: $P^2(\mathbb{R}^2, 1) \subset \mathbb{R}^{10}$ parametrized by $(x(0), y(0), x, y, X_{01}, Y_{10}, X_{11}, X_{02}, Y_{11}, Y_{20})$

$$\begin{aligned} X_{ij}(t) &= \int_0^t x(t)^i y(t)^j dx(t), \quad i+j \le 2, j \ne 0 \\ Y_{ij}(t) &= \int_0^t x(t)^i y(t)^j dy(t), \quad i+j \le 2, i \ne 0 \end{aligned}$$

Integration by parts relations:

$$X_{01} = xy - x(0)y(0) - Y_{10},\ X_{11} = \frac{1}{2}\left(x^2 y - x(0)^2 y(0) - Y_{20}\right)$$

$$X_{02} = xy^2 - x(0)y(0)^2 - 2Y_{11}$$

define $P^2(\mathbb{R}^2, 1) \subset \mathbb{R}^{10}$.

Translation-invariant substitution $X = x - x(0)$, $Y = y - y(0)$

$$\begin{aligned} \overline{X} &= a_{11}X + a_{12}Y \\ \overline{Y} &= a_{21}X + a_{22}Y \end{aligned}$$

$$\begin{aligned} \overline{Y_{10}} &= \int_0^t \overline{X} d\overline{Y} = Y_{10} + \frac{1}{2}a_{11}a_{21}X^2 + \frac{1}{2}a_{12}a_{22}Y^2 + a_{12}a_{21}XY \\ \overline{Y_{11}} &= \int_0^t \overline{XY} d\overline{Y} = a_{22}Y_{11} - a_{21}X_{11} \\ &+ \frac{1}{3}a_{21}{}^2 a_{11}X^3 + a_{21}a_{12}a_{22}XY^2 + a_{22}a_{11}a_{21}X^2Y + \frac{1}{3}a_{22}^2 a_{12}Y^3 \\ \overline{X_{11}} &= \int_0^t \overline{XY} d\overline{X} = a_{11}X_{11} - a_{12}Y_{11} \\ &+ \frac{1}{3}a_{11}^2 a_{21}X^3 + a_{12}a_{11}a_{21}X^2Y + a_{11}a_{12}a_{22}XY^2 + \frac{1}{3}a_{12}^2 a_{22}Y^3 \end{aligned}$$

$A(2) = GL(2) \ltimes \mathbb{R}^2$-invariants on $P^2(\mathbb{R}^2, 1) \Longleftrightarrow GL(2)$-invariants on $\mathbb{R}^5$ under the above action.

generally $H \ltimes \mathbb{R}^n$-invariants on $P^l(\mathbb{R}^n, 1) \Leftrightarrow H$-invariants $P^l(\mathbb{R}^n, 1)\backslash\mathbb{R}^n$

Computing Invariants

Smooth construction $\Rightarrow$ fundamental set of smooth local invariants:

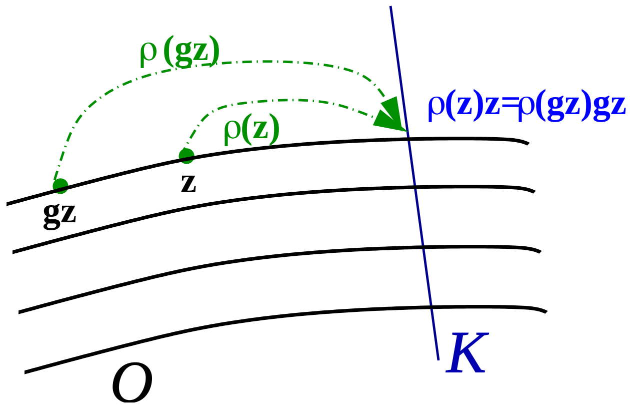
Fels, Olver (1999) (see also Cartan (1953), Griffiths(1974), Jensen (1977), Green (1978).)

Local cross-section $\mathcal{K}$ on $\mathcal{U} \subset M$ is a submanifold, s. t.

- $T_{\bar{z}}\mathcal{K}| \bigoplus T|_{\bar{z}}\mathcal{O}_{\bar{z}} = T|_{\bar{z}}M, \quad \forall \bar{z} \in \mathcal{K}$ (transversality condition)
- $\mathcal{K}$ intersects each connected component of $\mathcal{O}_{\bar{z}} \cap \mathcal{U}$ at the unique point.

$G \curvearrowright M$ semi. reg. (i.e. $\exists$s, $\forall \bar{z} \in M$ dim $\mathcal{O}_{\bar{z}} = s$) $\overset{\text{Frobenius Thm.}}{\Longrightarrow}$ $\forall z \in M \ \exists \mathcal{K} \ni z$.

Moving frame map $\rho : U \to G$ defined by $\rho(z) \cdot z \in \mathcal{K}$



$G \curvearrowright M$ free $\Rightarrow$ ρ is smooth, G-equivariant:

$$\rho(g \cdot z) \cdot (g \cdot z) = \rho(z) \cdot z \overset{\text{freeness}}{\Longrightarrow} \rho(g \cdot z) = \rho(z)\, g^{-1}$$

Invariantization: $\iota f(z) = f(\rho(z) \cdot z)$

Normalized invariants: $\iota z_1, \ldots, \iota z_n$ contains a set of fundamental invariants.

Example: $SE(2) = SO(2) \ltimes \mathbb{R}^2 \curvearrowright$ on jets plane curves $y = y(x)$:

Action: $\overline{x} = \cos(\phi)x - \sin(\phi)y + a, \quad \overline{y} = \sin(\phi)x + \cos(\phi)y + b$

Prolongation to derivatives up to second order:

$$\overline{y}_1 = \frac{\sin(\phi) + \cos(\phi)y_1}{\cos(\phi) - \sin(\phi)y_1}, \quad \overline{y}_2 = \frac{y_2}{(\cos(\phi) - \sin(\phi)y_1)^3}$$

Cross-section: $\mathcal{K} = \{(x, y, y_1, y_2) | x = 0,\, y = 0,\, y_1 = 0, y_2 > 0\} \subset \mathbb{R}^4$

Moving frame map is defined by: $\overline{x} = 0,\, \overline{y} = 0,\, \overline{y}_1 = 0 \Rightarrow$

$$\cos\phi = \frac{1}{\sqrt{y_1^2+1}},\ \sin\phi = -\frac{y_1}{\sqrt{1+y_1^2}}, \quad a = -\frac{x+y_1 y}{\sqrt{1+y_1^2}}, \quad b = \frac{y_1 x - y}{\sqrt{1+y_1^2}}.$$

Normalized invariants: $\iota x = 0,\ \iota y = 0, \iota y_1 = 0,$

$\iota y_2 = \frac{y_2}{(1+y_1^2)^{3/2}}$ is curvature –differential invariant.

Algebraic Approach ⇒ generators of rational invariants:

Hubert, Kogan (2007) (cf. Rosenlicht (1956), Vinberg, Popov (1989), Beth, Müller-Quade (1999))

Notation: $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $z = (x, y, y_1, y_2)$, $\overline{z} = (\overline{x}, \overline{y}, \overline{y}_1, \overline{y}_2)$

Group ideal: $G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{K}[\lambda]$

Action ideal: $J = \left(\overline{x} - (\lambda_1 x - \lambda_2 y + \lambda_3), \overline{y} - (\lambda_2 x + \lambda_1 y + \lambda_4), \overline{y}_1 - \frac{\lambda_2 + \lambda_1 y_1}{\lambda_1 - \lambda_2 y_1}, \overline{y}_2 - \frac{y_2}{(\lambda_1 - \lambda_2 y_1)^3}\right) + G \subset (\lambda_1 - \lambda_2 y_1)^{-1}\mathbb{K}[\lambda, z, \overline{z}]$

Cross-section ideal: $K = (\overline{x}, \overline{y}, \overline{y}_1)$

Graph-section ideal: $I^e = \left(\overline{x}, \overline{y}, \overline{y}_1, \overline{y}_2^2 - \frac{y_2^2}{(1+y_1^2)^3}\right) \subset \mathbb{K}(z)[\overline{z}]$, where $I = (J + K) \cap \mathbb{K}[z, \overline{z}]$. Coefficients of the reduced GB of I^e generate $\mathbb{K}(z)^G$

Replacement Invariants: $\overline{\mathbb{K}(z)^G}$-roots if I^e: $\xi_1^{\pm} = (0, 0, 0, \pm\kappa)$, where $\kappa = \frac{y_2}{(1+y_1^2)^{3/2}}$ is curvature.

Inductive construction for $G = B \cdot A \Rightarrow$ invariants of G in terms of invariants of A

Definition: G factors as a product of its subgroups A and B if $\forall g \in G$ $\exists a \in A$, $\exists b \in B$ s.t $g = ba$.

If $A \cap B = e$ then a and b are unique.

Proposition (K. 2003): If $G = B \cdot A$ and $\forall z \in M$ intersection of orbits $O(z)_A \cap O(z)_B$ is discrete then $\forall z \in M$ $\exists$ a loc. cross-section $\mathcal{K}_A \ni z$ invariant under the B- action.

Inductive variation of Fels-Olver construction:(K. 2003)

1. Restrict the G-action to A. Find B-invariant c.-s. $\mathcal{K}_A$.

2. Construct a moving frame map $\rho_A\colon M \rightarrow A$ defined by the condition $\rho_A(z)\cdot z \in \mathcal{K}_A$, $\forall z \in M$. Coordinates of the projection $\iota(z) = \rho_A(z)\cdot z\colon M \rightarrow \mathcal{K}_A$ are $A - invariant$.

3. Restrict the G-action to B action on $\mathcal{K}_A$ and choose c.-s. $\mathcal{K}_B \subset \mathcal{K}_A$.

4. Construct a moving frame map $\rho_B\colon \mathcal{K}_A \rightarrow B$ defined by the condition $\rho_B(z) \cdot z \in \mathcal{K}_B$, $\forall z \in \mathcal{K}_A$, by solving the corresponding equations.

5. The G-moving frame map $\rho\colon Z \rightarrow G$ is defined by $\rho(z) = \rho_B(\rho_A(z) \cdot s)\rho_A$, and G-invariants are coordinate components of $\rho(z) \cdot z = \rho_B(\rho_A(z) \cdot z) \cdot (\rho_A(z) \cdot z) = \rho_B(\iota_A(z)) \cdot \iota_A(z)$.

Example $n=2$ $SA(2) = B \cdot SE(2)$, where

$$B = \left\{ \begin{pmatrix} b_{11} & b_{12} \\ 0 & \frac{1}{b_{11}} \end{pmatrix} | b_{11} > 0 \right\}$$

$SE(2)$-invariants:(***)

$$X^E = \sqrt{X^2+Y^2}, \quad Y_{10}^E = Y_{10} - \frac{1}{2}XY,$$

$$Y_{11}^E = \frac{3\,XY_{11} + 3\,YX_{11} - 2\,Y^2X^2}{3\sqrt{X^2+Y^2}}, X_{11}^E = \frac{3\,XX_{11} - 3\,YY_{11} + XY^3 - YX^3}{3\sqrt{X^2+Y^2}}.$$

$SA(2)$-invariants:(***)

$$I_1 = Y_{10}^E = Y_{10} - \tfrac{1}{2}XY, \quad I_2 = X^E Y_{11}^E = XY_{11} + YX_{11} - \tfrac{2}{3}X^2Y^2$$

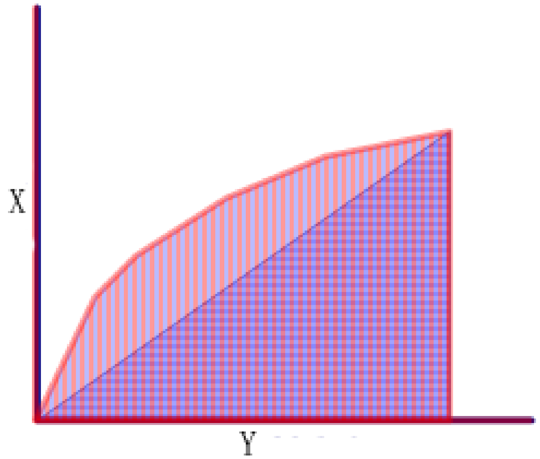
$A(2)$-invariant: $I = \frac{I_2}{I_1^2}$

(***) Substitute $X = x - x(0)$, $Y = y - y(0)$

Geometric interpretation

$\gamma(t) = (X(t), Y(t)) \subset \mathbb{R}^2$ is a curve such that $\gamma(0) = (0, 0)$.

$I_1(t) = \int_0^t X dY - \frac{1}{2} XY$
is the (signed) area between
the curve and the secant.



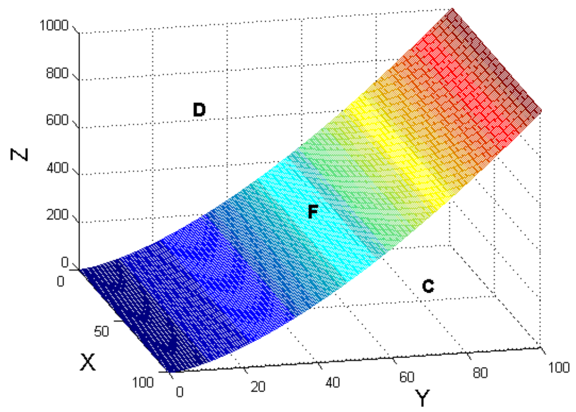
$$I_2(t) = -\frac{1}{3}((X^2Y^2 - 3X \int_0^t XY dY) + (X^2Y^2 - 3 \int_0^t XY dY))$$

Define $Z(t) = X(t)Y(t) \Longrightarrow \gamma(t)$ is lifted to 3D:
$\hat{\gamma}(t) = (X(t), Y(t), Z(t))$.

$$I_2 = -\frac{1}{3}((XYZ - 3X \int_0^t Z dY) + (XYZ - 3Y \int_0^t Z dX)).$$

$$I_2 = -\frac{1}{3}\left(\left(XYZ - 3X\int_0^t ZdY\right) + \left(XYZ - 3Y\int_0^t ZdX\right)\right).$$

$$\widehat{\gamma} = (X(t), Y(t), Z(t) = X(t)Y(t)) \Longrightarrow F = \widehat{\gamma}(t) \times [0, X(t)].$$



XYZ is signed volume of rectangular prism $[0, X(t)] \times [0, Y(t)] \times [0, Z(t)]$.

$\int_0^t ZdY$ is the (signed) area "under" the projection of $\widehat{\gamma}(t)$ to the YZ-plane

$X\int_0^t ZdY$ is the signed volume C "under" the surface F

SA(3)-invariants for curves in 3D

$$
\begin{aligned}
I_1 &= n_1 X + n_2 Z - n_3 Y \\
I_2 &= 2\,n_1(XYZ^2 - 3\,Z_{011}X + 3\,YZ_{101} \\
&- ZZ_{110} - 2\,ZY_{101}) + n_2(2\,XY^2Z + 3\,XZ_{020} \\
&- 6\,ZX_{020} - 4\,YZ_{110} - 2\,YY_{101}) - 2\,n_3(3\,YX_{101} \\
&- 3\,ZX_{110} + XZ_{110} - XY_{101})
\end{aligned}
$$

where

$$n_1 = \frac{YZ}{2} - Z_{010}, n_2 = \frac{XY}{2} - Y_{100}, n_3 = \frac{XZ}{2} - Z_{100}.$$

$$X_{ijk} = \int_0^t X^i Y^j Z^k dX, \text{ etc.}$$

A(3)-invariants for curves in 3D

The quotient $I = \frac{I_2}{I_1^2}$

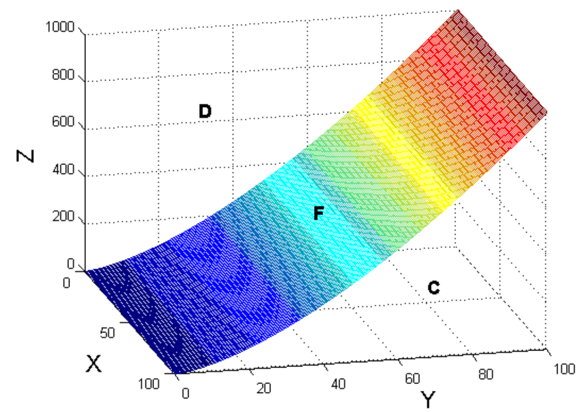
Geometric interpretation

$\gamma(t) = (X(t), Y(t), Z(t)) \subset \mathbb{R}^3$ is a curve s.t $\gamma(0) = (0, 0, 0)$.

$$I_1(t) = n_1(t)X(t) + n_2(t)Z(t) - n_3(t)Y(t)$$

$$n_1(t)\, X(t) = X\left(\tfrac{1}{2}YZ - \textstyle\int_0^t Y\, dZ\right)$$

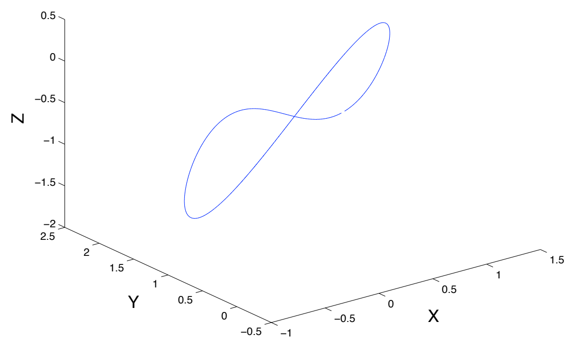
is the (signed) volume C
"under" the surface $F = \gamma(t) \times [0, X(t)]$.



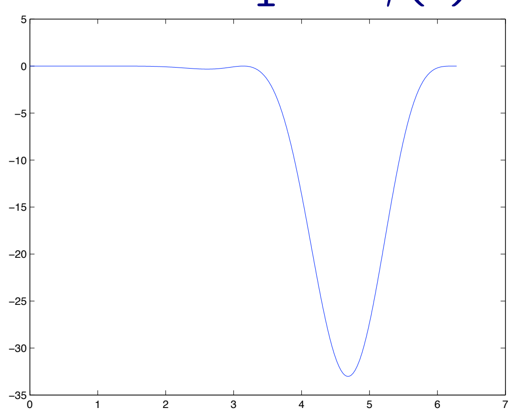
I_2–???

Example:

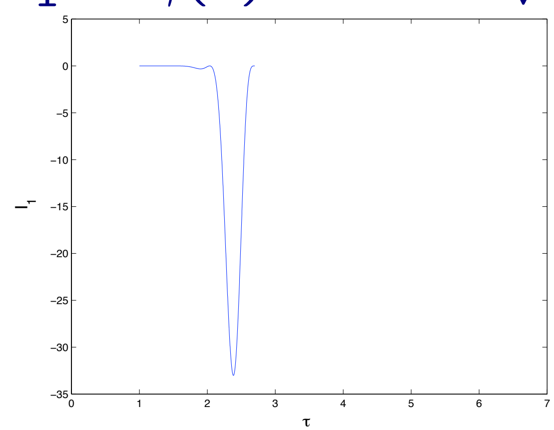
$$\gamma(t) = (\sin t - \frac{1}{5}\cos^2 t + \frac{1}{5}, \frac{1}{2}\sin t - \cos t + 1\ \sin^2 t + \cos t - 1)$$



I_1 for $\gamma(t)$

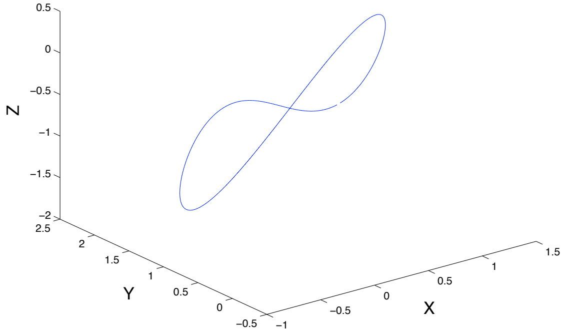


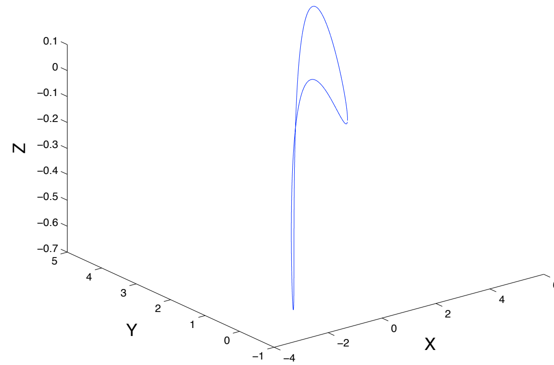
I_1 for $\gamma(\tau)$ with $\tau = \sqrt{t+1}$



$\gamma(t)$

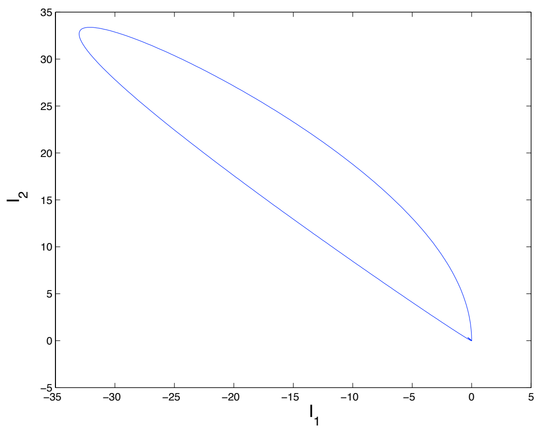
$\bar{\gamma}(t) = g \cdot \gamma,\ g \in SA(3).$



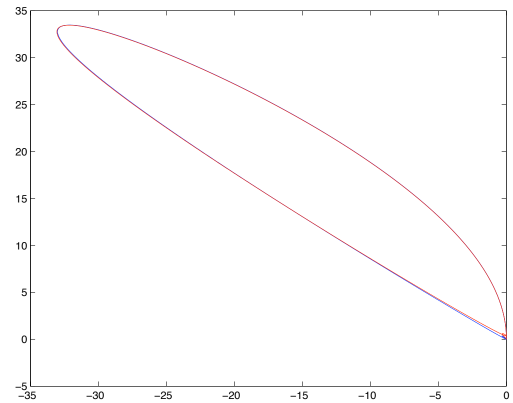


signatures for γ and $\bar{\gamma}$

continuous



discrete



Signatures are independent of parametrization

What is next?

Numerics: Noise sensitivity, comparison with other types of invariants, such as joint invariants.

Applications: Image recognition/classifications

Theory of integral invariants:

- Structure of integral invariants (cf. finitely generated differential algebra of invariants)
- Signatures:
  - Discrimination power of signatures based on two invariants.
  - Symmetry detection.

Projective and other groups...

Comparison with Euclidean signatures defined by differential invariants

Sine and Cosine	
$u = \cos(x)$	$u = \sin(x)$
$\kappa = -\dfrac{\cos(x)}{(1+\sin^2(x))^{(3/2)}}$	$\kappa = \dfrac{\sin(x)}{(1+\cos^2(x))^{(3/2)}}$
$\kappa_s = \dfrac{\sin(x)(1+\cos^2(x))}{(1+\sin^2(x))^3}$	$\kappa_s = -\dfrac{\cos(x)(1+\sin^2(x))}{(1+\cos^2(x))^3}$