A story of two postulates

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SWiM, June 22, 2018.

Euclid of Alexandria

Mid-4th century BC - Mid-3th century BC
350 - 250 BC

Geometry that we learn at school is called Euclidean geometry.

It is based on the Euclidean parallel postulate

Why is this man so famous?

What is a postulate?

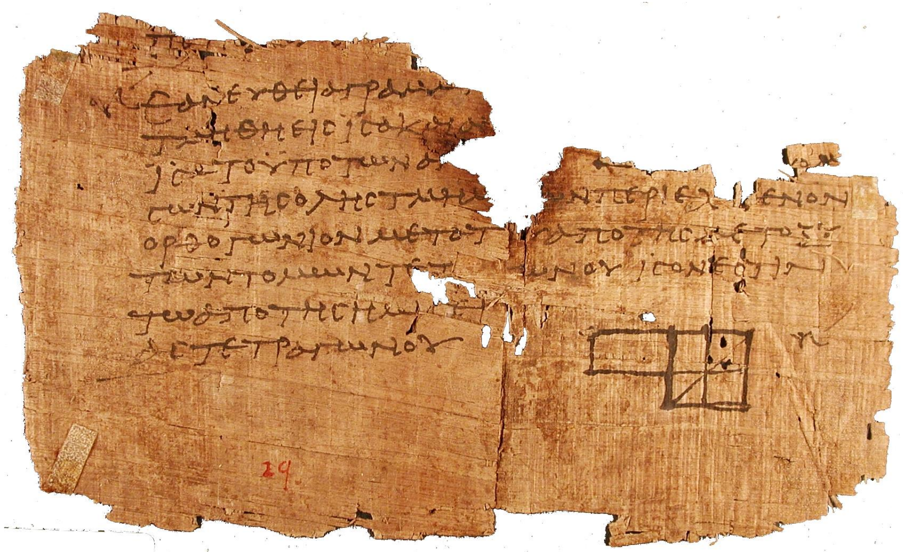
How did Euclid look like?

from Wikipedia (source ?)

Raffaello Sanzio (Raphael), c. 1510, Palazzi Pontifici, Vatican

Jusepe de Ribera, c. 1630-1635, J. Paul Getty Museum

Euclid’s Elements circa 300 BC.



A fragment of Euclid’s Elements on part of the Oxyrhynchus papyri $\sim$ 100 AD

The earliest preserved complete version: c. 850 AD,
the Vatican Library

First English version, 1570

Elements consist of 13 books containing 465 Propositions about planar and 3D geometry, as well as number theory. This includes:

- Most of the theorems for planar geometry we learn at school.

- Euclid's algorithm for finding the greatest common divisor and the least common multiple.

- Proof is irrationality of the square roots of non-square integers (e.g. $\sqrt{2}$)

- Volumes of cones, pyramids, and cylinders in detail by using the method of exhaustion, a precursor to integration.

Most of the results are not original!

According to W.W. Rouse Ball,“ A Short Account of the History of Mathematics”, 1908

- Pythagoras (c. 570 - 495 BC) was probably the source for most of books I and II;
- Hippocrates of Chios (c. 470 - 410 BC) for book III;
- Eudoxus of Cnidus (c. 408 - 355 BC) for book V, while books IV, VI, XI, and XII probably came from other Pythagorean or Athenian mathematicians.

... and still it is one of the most published and influential books!

Elements provided a well organized exposition of a large body of mathematics

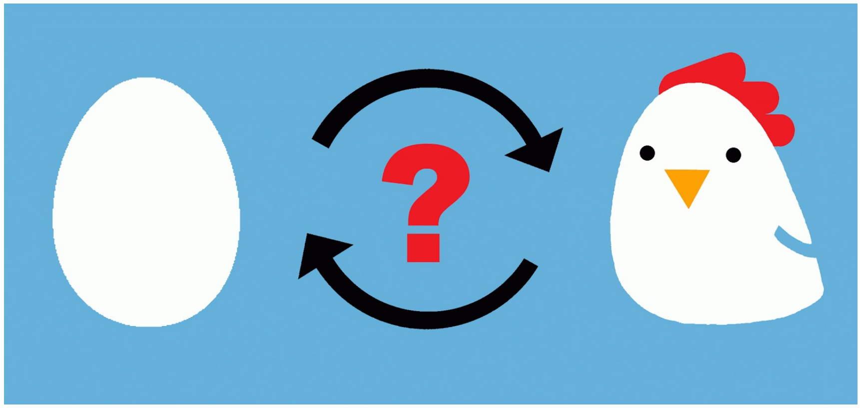
Served as text-book since it was written until 19 th century!

Elements set up the modern standard of a mathematical theory

- Each new statement must be proved - deduced from the previously proved statements using logic.

- Wait... but, how is the first statement deduced?

Do we have the chicken and egg problem ?



The first statements, called postulates, or axioms, have to be accepted without a proof.

A structure of a mathematical theory:

$\boxed{\text{Postulates}}$ (or axioms) are statements accepted without a proof. They form the $\boxed{\text{foundation of a theory}}$.

$\boxed{\text{Theorems}}$ (or propositions, or lemmas, or corollaries) are statements proved from the postulates and previously proved theorems using logic. They form the $\boxed{\text{body of a theory}}$.

A good system of postulates is

- consistent - does not lead to a contradiction.

- complete - no other unstated assumptions are needed to carry out the proofs.

- is non-redundant - does not contain postulates that can be proved using the others.

Postulates as stated by Euclid circa 300BC
(translated by Sir Thomas Heath)

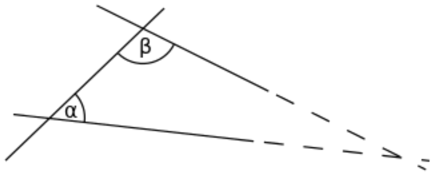
I. To draw a straight line from any point to any point.

II. To produce a finite straight line continuously on a straight line.

III. To describe circle in any center and radius.

IV. That all right angles equal one another.

V. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced infinitely, meet on that side on which the angles are less than the two right angles.



$\boxed{\alpha + \beta < 180}$

Remarks

1. Euclid's system is not complete: he uses more unproved statements than he stated as postulates. Modern systems of postulates substitute I. - IV. with a larger number postulates.

   David Hilbert (late 19th century German mathematician) replaced I. - IV. with 19 statements.

   From now on by Euclid's I. - IV., I mean a completed version of I. - IV.

2. Postulate V. (called the parallel postulate) looks very different from the other four.

   - It is much more involved - looks like a theorem

   - Euclid refrained from using it if he can base a proof on I. - IV. (Proposition 1 - 28 in Book I are proved without it)

$\boxed{\text{Why is V. called the parallel postulate ?}}$

Definition: Two lines in the plane are parallel

if they have no common points.

Assuming Euclid's I. - IV., one can show that V. is equivalent to

For every line l and for every external point P, there exists $\boxed{\textit{exactly one line}}$ *m such that P lies on m and m is parallel to l.*

Is Euclid's V-th postulate a theorem?

Attempts to prove the V-th postulate started in ancient Greece and continued by Arab and European mathematicians until late 19-th century

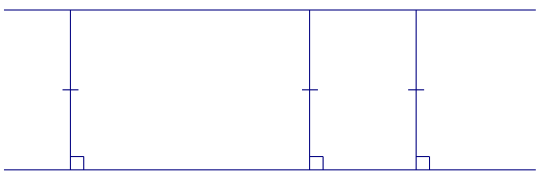
- changing the definition of parallel lines.
- accepting another "self-evident" statement which is actually equivalent to the V-th postulate.

Examples of such attempts

Posidonius (c.135 - 51 BC) - Greek philosopher, politician, astronomer, geographer and historian.

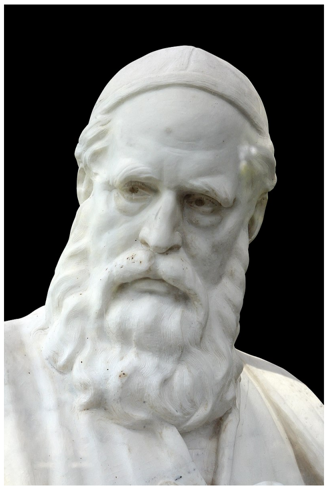
c. 60 BC Naples, National Archaeological Museum

Proposed to call two coplanar lines parallel if they are equidistant:



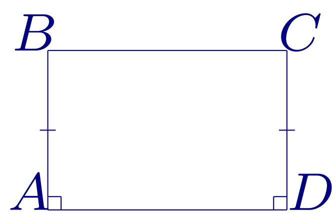
$\boxed{\text{Existence of equidistant lines} \Longleftrightarrow \text{Euclid's V. postulate}}$

Omar Khayyám (1048 - 1131) - Persian mathematician, astronomer and poet.



Bust by Sadighi (c. 1960) in Nishapur, Iran.

In *“Discussion of Difficulties in Euclid”*, considered what is now called Khayyám - Saccheri quadrilateral:



1. $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$, $\overleftrightarrow{DC} \perp \overleftrightarrow{AD}$
2. $\overline{AB} \cong \overline{DC}$

- $\boxed{\text{Summit angles are right} \iff \text{Euclid's V. postulate}}$
- studied the acute and obtuse cases, proved many correct results but refuted them as contradictory to basic principles.

John Wallis (1616-1703) - English clergyman and mathematician, served as chief cryptographer for the parliament and the royal court.

by Kneller (1701), University of Oxford collection.

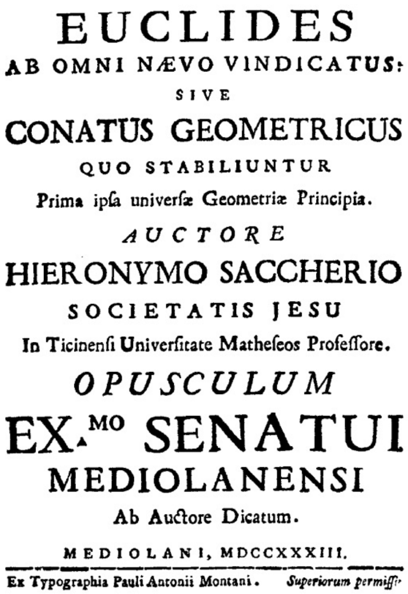
Based his proof on an assumption:

"For every figure there exists a similar figure of arbitrary magnitude."

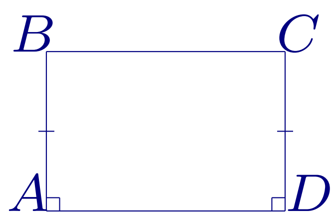
(cited from Roberto Bonola, Non-Euclidean Geometry, 1912.)

Existence of similar but not congruent figures $\Longleftrightarrow$ Euclid's V. postulate

Giovanni Saccheri (1667–1733) - Italian Jesuit priest, philosopher and mathematician.


EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permissu.

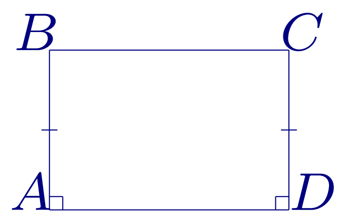
"Euclide Ab Omni Naevo Vindicatus" (1733)
["Euclid Freed of Every Flaw"]



- Showed correctly that if summit angles are obtuse there is a contradiction with Euclid's II postulate: "To produce a finite straight line continuously on a straight line".

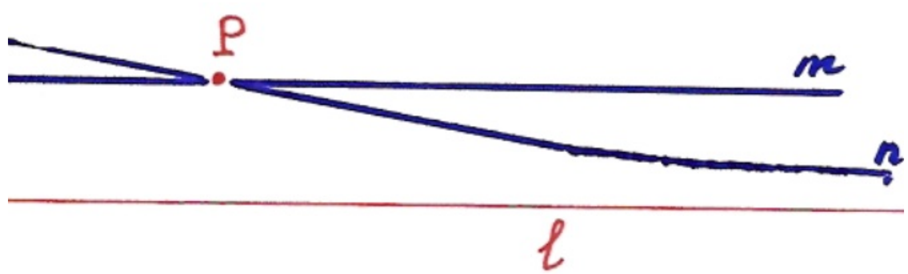
- Through a very involved and incorrect argument showed that acute case leads to a contradiction.

In modern terms:



- The obtuse case corresponds to the elliptic geometry, where there are no parallel lines.

- The acute case corresponds to the hyperbolic geometry, where there is more than one parallel line to a given line through a given external



  point.

Saccheri proved many valid results of elliptic and hyperbolic geometries, however he did not accept these geometries as possible alternatives!

Nikolai Lobachevsky (1792 - 1856) a Russian mathematician
and
János Bolyai (1802 - 1860) a Hungarian mathematician

were brave enough to accept those as valid possibilities, study them deeply and to publish their results.

Carl Friedrich Gauss (1777 - 1855) a German mathematician and physicist (king of geometry) expressed similar ideas (not in details) in private correspondence, but did not make make public for the fear of controversy.

Gauss to Bessel in 1829:

It may take very long before I make public my investigation on this issue; In fact this may not happen in my lifetime for I fear the scream of the dullards if I made my views explicit.

(cited from Herbert Meschkowski, Non-Euclidean Geometry, 1964, p. 33)

Nikolai Lobachevsky (1792 - 1856)

by Lev Krioukov, 1839

ВООБРАЖАЕМАЯ

ГЕОМЕТРІЯ.

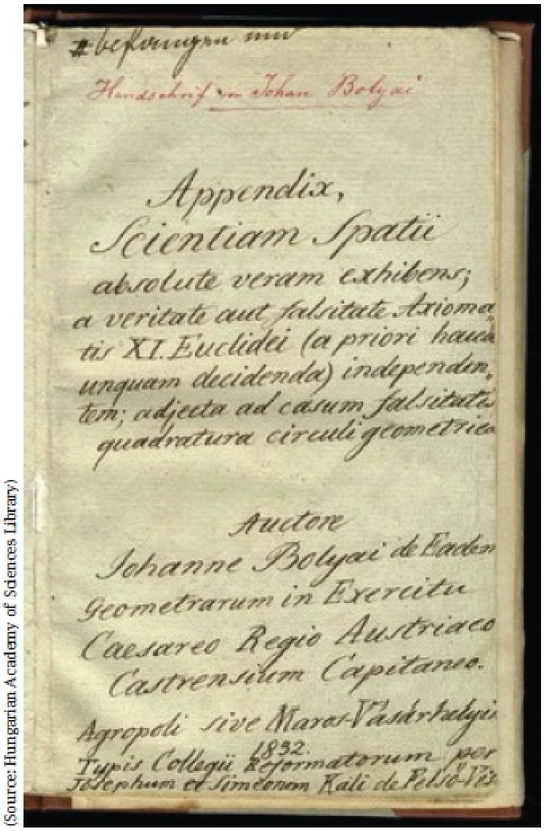
КАЗАНЬ.

1835.

“Imaginary Geometry” 1835 (presented first in 1826)

János Bolyai (1802 - 1860)

Relief on Culture Palace in Marosvásárhely (1911-1913)



Appendix,
Scientiam Spatii
absolute veram exhibens;
a veritate aut falsitate Axiomatis XI. Euclidei (a priori haud unquam decidenda) independentem; adjecta ad casum falsitatis quadratura circuli geometrica.

Auctore
Johanne Bolyai de Eadem
Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

Agropoli sive Maros-Vásárhelyini
1832.
Typis Collegii Reformatorum per Josephum et Simeonem Kali de Felső-Vist.

(Source: Hungarian Academy of Sciences Library)

1832

“The absolute true science of space” Appendix to “Tentamen juventutem studiosam in elements Matheseos” by Farkas Bolyai

Wrong images of János Bolyai

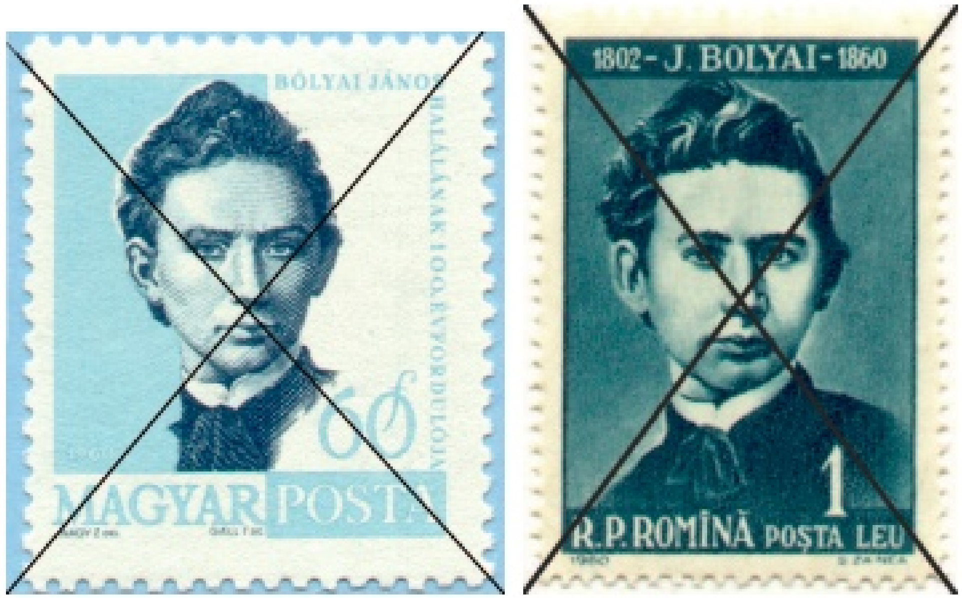


Figure 2. The portrait, not of János Bolyai (on Hungarian and Romanian stamps in 1960), that has been circulated around the world.

“Real Face of János Bolyai ” by Tamás Dénes in AMS Notices 2011. `https://www.ams.org/notices/201101/rtx110100041p.pdf`

Some remarkable quotes

Farkas Bolyai to János Bolyai:

I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallel alone.

(cited from Herbert Meschkowski, Non-Euclidean Geometry, 1964. p. 31.)

János Bolyai to Farkas Bolyai on November 3, 1823:

I am now resolved to publish a work on the theory of parallels. ... I created a new, different world out of nothing.

(cited from Herbert Meschkowski, Non-Euclidean Geometry, 1964, p. 98)

Carl Friedrich Gauss to Farkas Bolyai on reading the Appendix in 1832:

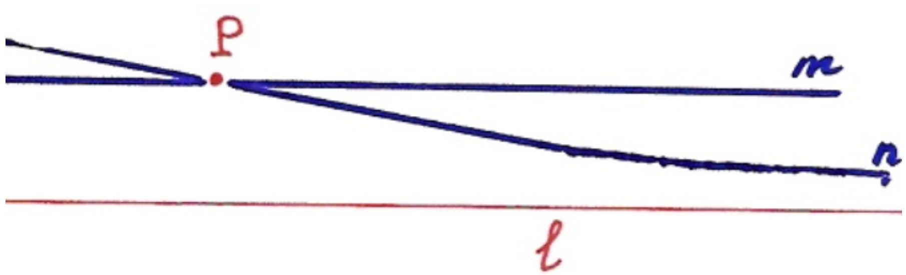
To praise it would amount to praising myself. For the entire content of the work...coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years....It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that is just the son of my old friend, who takes the precedence of me in such remarkable manner.

(cited from Roberto Bonola, Non-Euclidean Geometry, Dover edition,1955, p.100 (original book was published in 1912).)

Hyperbolic geometry.

is based on (a completion of) Euclid's postulates I.-IV.
and V. is replaced with:

$\tilde{\text{V}}$. For every line l and for every external point P, there are $\boxed{\text{at least two lines}}$ m and n such that $P \in m$, $P \in n$, $m \,||\, l$ and $n \,||\, l$.



In the hyperbolic geometry:

- The sum of the measures of angles in any triangle is $< 180^\circ$.
- There are no rectangles.
- There are no equidistant lines.
- The summit angles in Saccheri quadrilaterals are acute.
- All similar figures are congruent.
- There is a bound on areas of triangles.
- Trigonometry is different from the Euclidean one ($\cosh$ and $\sinh$).

How can one visualize this unusual world?

Lobachevsky and Bolyai did not offer any models for this unusual geometry.

What if there is still a contradiction that was not yet detected?

Models were created by

- Eugenio Beltrami (1835 - 1900) an Italian mathematician.
- Felix Klein (1849 - 1925) a German mathematician.
- Henri Poincaré (1854 - 1912) a French mathematician.

From then on, the word “geometry” became plural...

$\Gamma \varepsilon \omega \mu \varepsilon \tau \rho \acute{\iota} \alpha \varsigma$

Euclidean Geometry

Hyperbolic Geometry | Elliptic Geometry | Riemannian Geometry

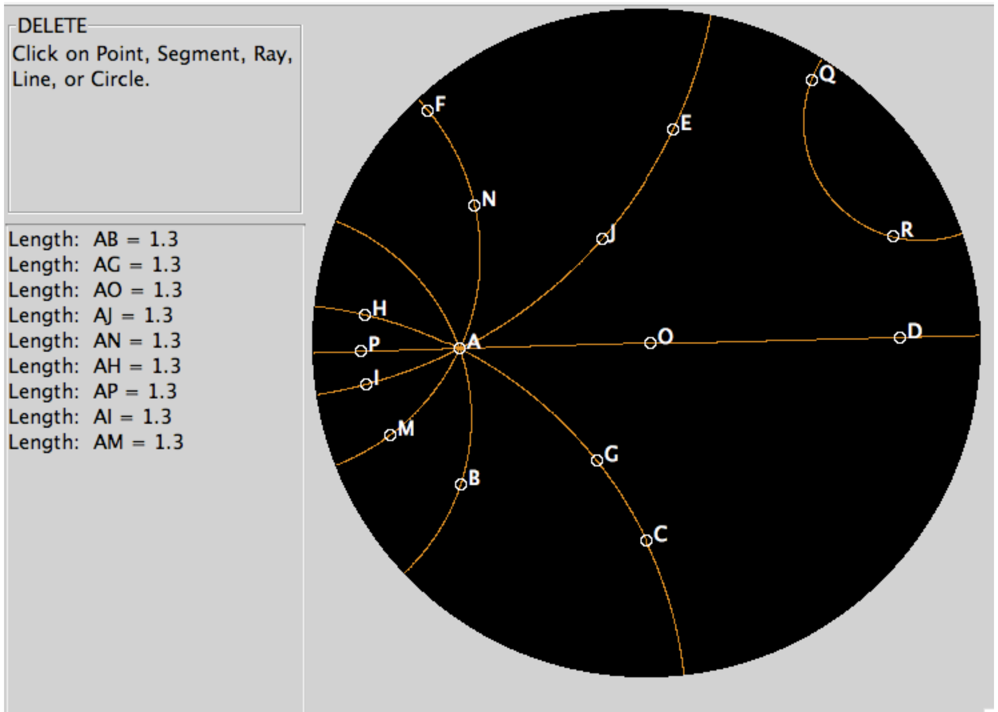
Pseudo-Riemannian Geometry | Affine Geometry | Projective Geometry

Finite Geometry | Finsler Geometry | Symplectic Geometry

Poincaré Disk Model of Hyperbolic Geometry

with
NonEuclid Java Software

http://www.cs.unm.edu/~joel/NonEuclid/NonEuclid.html

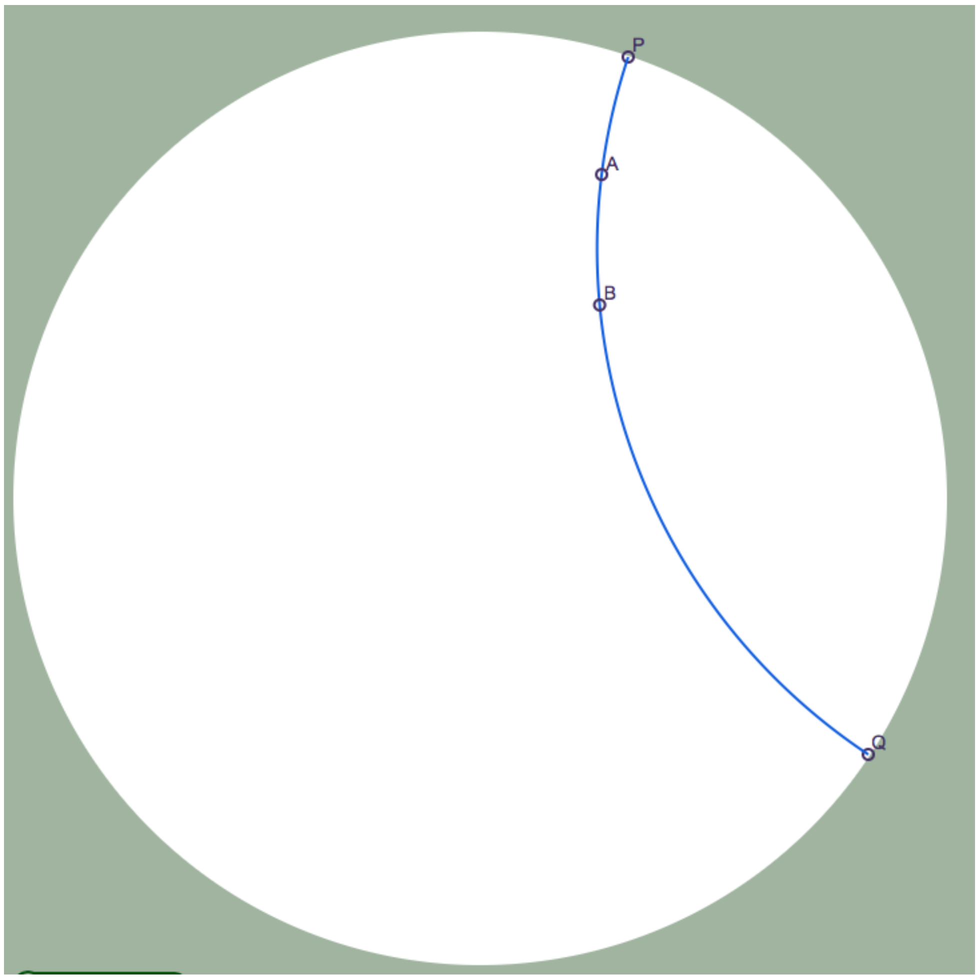


Demonstration 1 with NonEuclid

- Draw a line through any two points.
- Through a point not on the first line, draw several parallel lines
- Demonstrate two limiting parallel lines.
- Discuss that using the Euclidean distance in this model would violate Euclid's second postulate.

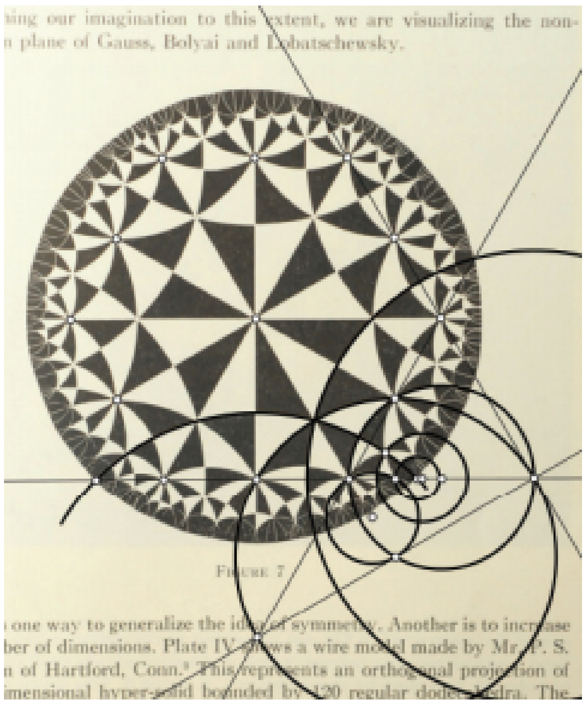
Distance in the Poincare Disk

Euclid's II postulate:
To produce a finite straight line continuously on a straight line

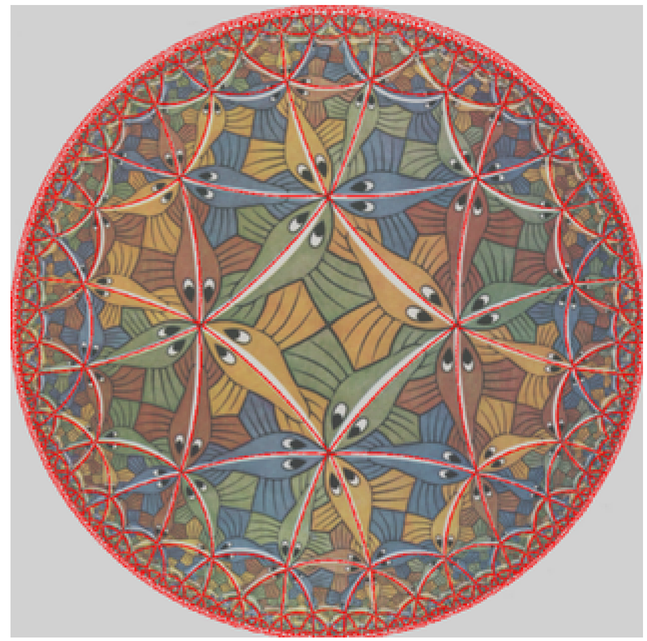


$$\mathrm{Dist}(A,B) = \left| \ln\left(\frac{|AP| \cdot |BQ|}{|AQ| \cdot |BP|} \right) \right|$$

Tessellations by Harold Coxeter and Maurits Cornelis Escher*



Coxeter's tessellation 90°-45°-30° triangles from Trans. Royal Soc. Canada (1957), with Escher's markings.



Escher's Circle Limit III. 1959

*"The Mathematical Side of M. C. Escher" by Doris Schattschneider in AMS Notices 2010. `https://www.ams.org/notices/201006/rtx100600706p.pdf`

Demonstration 2 with NonEuclid

- Draw an equilateral triangle with sides 1 using Euclid's construction with two circles. Measure its angles. Move it around to see how its shape visually changes, while the measurements remain the same.

- Show how the Wallis postulate fails by trying to construct an equilateral triangle with sides 2 and the same angles. Discuss that in Hyperbolic geometry angles of a triangle determine its size. (Intrinsic notion of length). Compare with Euclidean geometry.

Demonstration 3 with NonEuclid

- Construct a Saccheri quadrilateral.
- Is it a parallelogram? Is it a rectangle? (check, measure)
- Do the summit and the base have a common perpendicular? If yes, how many?
- What is longer: the summit vs. the base; the common perpendicular vs. the sides?