# Object-image correspondence for curves under central and parallel projections

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# Problem and motivation:

**Given:** A a subset $\mathfrak{S} \subset \mathbb{R}^n$ and a subset $\mathfrak{s} \subset \mathbb{R}^{n-1}$ $(n \geq 3)$ and a class of admissible maps from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$;

**Decide:** whether $\mathfrak{s}$ is the image of $\mathfrak{S}$ under a projection from this class.

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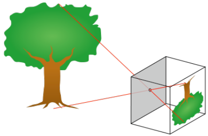
**Motivation:** Establishing a correspondence between objects in 3D and their images, *when camera parameters and position are unknown*.

**Focus:**
- $n = 3$;
- central and parallel projections from $\mathbb{R}^3$ to a plane in $\mathbb{R}^3$.
- objects are curves in $\mathbb{R}^3$ and $\mathbb{R}^2$
- finite sets of points will be also discussed

**Generalization:** to $n > 3$ is theoretically straightforward but computationally more challenging

# Pinhole camera:



Let $(z_1, z_2, z_3)$ be the standard coordinates in $\mathbb{R}^3$ and assume:

- camera is located at $(0, 0, 0)$;
- image plane passes through $(0, 0, 1)$ and is perpendicular to $z_3$-axis;
- coordinates on the image plane correspond to $(z_1, z_2)$.

A point $(z_1, z_2, z_3)$ with $z_3 \neq 0$ is projected to the point

$$(x, y) = \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}\right).$$

# General camera models:

Degrees of freedom:

- location of the projection center (3 degrees of freedom);
- the position of the image plane (3 degrees of freedom)
- choice of, in general, non-orthogonal, coordinates on the image plane (5 degrees of freedom, since the overall scale is absorbed by the choice of the distance between the image plane and the camera center).

$$\Downarrow$$

11 real parameters $[p_{ij}]_{j=1\dots4}^{i=1\dots3}$ (equivalent under scaling $p_{ij} \to \lambda p_{ij}$, $\lambda \neq 0$.)

$$\begin{aligned} x &= \frac{p_{11}\, z_1 + p_{12}\, z_2 + p_{13}\, z_3 + p_{14}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}}, \\ y &= \frac{p_{21}\, z_1 + p_{22}\, z_2 + p_{23}\, z_3 + p_{24}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}}. \end{aligned}$$

# Homogeneous coordinates:

- [] around matrices (and, in particular, vectors) denotes an equivalence class with respect to multiplication of a matrix by a nonzero scalar.
- For matrices $A$ and $B$ of appropriate sizes $[A]\,[B] := [A\,B]$.
- $\mathbb{R}^n \hookrightarrow \mathbb{P}^n$: $(z_1, \ldots, z_n) \to [z_1, \ldots, z_n, 1]$.
- points at infinity are $[z_1, \ldots, z_n, 0]$, where $\exists i = 1, \ldots, n : \; z_i \neq 0$.

# Camera models in homogeneous coordinates:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix}$$

- $P$ is $3 \times 4$ matrix of rank 3
- $\exists$ non-zero point $(z_1^0, z_2^0, z_3^0, z_4^0) \in \mathbb{R}^4$ s. t. $P\,(z_1^0, z_2^0, z_3^0, z_4^0)^{tr} = (0, 0, 0)^{tr}$.
- $[z_1^0, z_2^0, z_3^0, z_4^0] \in \mathbb{P}^3$ is the center of projection.

# Types of projections:

Projection (camera) is called

- **finite ($\mathcal{FP}$)** if its center is not at $\infty \Leftrightarrow$ left $3 \times 3$ submatrix of $P$ is non-singular (central projection with 11 degrees of freedom);
- **infinite** center is at $\infty$;
- **affine ($\mathcal{AP}$)** center is at $\infty$ and the preimage of the line at $\infty$ in $\mathbb{P}^2$ is the plane at infinity in $\mathbb{P}^3 \Leftrightarrow$ the last row of $P$ is $[0, 0, 0, \lambda]$, $\lambda \neq 0$ (parallel projection with 8 degrees of freedom).

# Standard projections:

- Standard finite projection (simple pinhole camera)

$$P_f^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Standard affine projection (orthogonal projection on $z_1 z_2$-plane)

$$P_a^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

# Projection problem for curves

Set up and notation:

- $\gamma\colon I_\gamma \to \mathbb{R}^n$, where $I_\gamma \subset \mathbb{R}$ is an interval, be a smooth parametric curve.
- $C_\gamma := \{\gamma(t) | t \in I_\gamma\}$ its image in $\mathbb{R}^n$.
- $[C_\gamma]$ is the corresponding set of points in $\mathbb{P}^n$ (represented in homogenous coordinates as column $(n+1)$-vectors)

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Problem:

**given** $\Gamma\colon I_\Gamma \to \mathbb{R}^3$ and $\gamma\colon I_\gamma \to \mathbb{R}^2$

**decide** if there exists a finite projection $[P] \in \mathcal{FP}$ (or an affine projection $[P] \in \mathcal{AP}$) such that

$$[C_\gamma] = [P]\,[C_\Gamma]$$

# Example

$$\Gamma(s) = (z_1(s),\, z_2(s),\, z_3(s)) = \left(s^3,\, s^2,\, s\right),\ s \in \mathbb{R}$$

can be projected to

- $\gamma_1(t) = \left(t^2,\, t\right),\ t \in \mathbb{R}$ by the standard finite projection

$$P_f^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- and to $\gamma_2(t) = \left(\frac{t^3}{t+1},\, \frac{t^2}{t+1}\right),\ t \in \mathbb{R}$ by finite projection

$$P := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

- but not to $\gamma_3(t) = \left(t,\, t^5\right),\ t \in \mathbb{R}$.

# Groups:

- projective: $\mathcal{PGL}(n+1) = \{[B] | B \in \mathcal{G}L(n+1)\}$
- affine $\mathcal{A}(n) = \{[B] | B \in \mathcal{G}L(n+1)$, the last row of $B$ is $(0, \dots, 0, 1)\}$.
- special affine $\mathcal{SA}(n) = \{[B] | B \in \mathcal{S}L(n+1)$, the last row of $B$ is $(0, \dots, 0, 1)\}$.

$\mathcal{PGL}(n+1)$ acts on $\mathbb{P}^n$ by
$([B], [z_1, \dots, z_n, z_0]^{tr}) \to [B]\,[z_1, \dots, z_n, z_0]^{tr}$.

$$\Downarrow$$

$\mathcal{PGL}(n+1)$ acts on $\mathbb{R}^n$ by linear fraction transformations.

# Action on cameras:

- $\mathcal{PGL}(3) \times \mathcal{A}(3)$ acts on the set $\mathcal{FP}$ of finite projections.
- $\mathcal{A}(2) \times \mathcal{A}(3)$ acts on the set $\mathcal{AP}$ of affine projections.

$$([A], [B]), [P]) \rightarrow [A]\,[P]\,[B^{-1}].$$

Both actions are transitive $\Rightarrow \mathcal{FP}$ and $\mathcal{AP}$ are homogeneous spaces.

## Proof of the transitivity of $\mathcal{PGL}(3) \times \mathcal{A}(3)$-action on $\mathcal{FP}$:

- $\forall [P] \in \mathcal{FP}$: $P$ is $3 \times 4$ matrix whose left $3 \times 3$ submatrix is non-singular. $\Rightarrow \exists\ c_1,\ c_2,\ c_3 \in \mathbb{R}$ s. t. $p_{*4} = c_1\, p_{*2} + c_2\, p_{*2} + c_3\, p_{*3}$, where $p_{*j}$ is the $j$-th column of $P$.
- Define $A$ to be the left $3 \times 3$ submatrix of $P$ and
$$B := \begin{pmatrix} 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
- Observe that $[A] \in \mathcal{PGL}(3)$, $[B] \in \mathcal{A}(3)$ and
$$P = A\, P_f^0\, B^{-1}, \text{ where } P_f^0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

# 9-dim'l. stabilizer of $P_f^0$:

$$H_f^0 = \left\{ \left( [A], \begin{bmatrix} A & \mathbf{0}^{\mathbf{tr}} \\ \mathbf{0} & 1 \end{bmatrix} \right) \right\}, \text{ where } A \in \mathcal{G}L(3).$$

# Corollary. Projection criterion for central projections (finite cameras):

A spatial curve $\Gamma(s) = \big(z_1(s),\, z_2(s), z_3(s)\big)$, $s \in I_\Gamma$ projects onto a planar curve $\gamma(t) = \big(x(t), y(t)\big)$, $t \in I_\gamma$, by *a central projection*

$$\Updownarrow$$

$\exists\ c_1, c_2, c_3 \in \mathbb{R}$ such that planar curves $\gamma(t)$ and

$$\epsilon_{c_1,c_2,c_3}(s) = \left(\frac{z_1(s) + c_1}{z_3(s) + c_3}, \frac{z_2(s) + c_2}{z_3(s) + c_3}\right)$$

are $\mathcal{PGL}(3)$-equivalent:

$$\exists [A] \in \mathcal{PGL}(3),\ \text{s. t.}\, C_\gamma = [A] \cdot C_{\epsilon_{c_1,c_2,c_3}}.$$

# Projection criterion for parallel projections (affine cameras):

A spatial curve $\Gamma(s) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in I_\Gamma$ projects onto a planar curve $\gamma(t) = \big(x(t), y(t)\big)$, $t \in I_\gamma$, by *a prallel projection*

$$\Updownarrow$$

$\exists\ b, c, f \in \mathbb{R}$ such that planar curves $\gamma(t)$ is $\mathcal{A}(2)$-equivalent to one of the following curves:

$$\begin{aligned}
\alpha(s) &= (z_2(s), z_3(s)),\\
\beta_b(s) &= \big(z_1(s) + b\, z_2(s),\ z_3(s)\big),\\
\delta_{cf}(s) &= \big(z_1(s) + c\, z_3(s),\ z_2(s) + f\, z_3(s)\big).
\end{aligned}$$

# A group-equivalence problem for curves with free parameters!

A solutions based on differential signature.

# Differential invariants for planar curves $\gamma(t) = (x(t), y(t))$:

- $G$ is an $r$-dim'l Lie group acting on the plane.
- For almost all actions $\exists$ two differential invariants $J_G$ of $K_G$ differential order $r-1$ and $r$ respectively.

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- Euclidean: $\kappa = \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$ and $\kappa_s$, where $ds = \sqrt{\dot{x}^2+\dot{y}^2}\,dt$
- equi-affine ($\mathcal{SA}(3)$): $\mu = \frac{3\,\kappa\,(\kappa_{ss}+3\,\kappa^3)-5\,\kappa_s^2}{9\,\kappa^{8/3}}$ and $\mu_\alpha$, where $d\alpha = \kappa^{1/3}ds$.
- projective ($\mathcal{PGL}(3)$): $\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha - 7\,\mu_{\alpha\alpha}^2 - 9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}$ and $\mu_\rho$, where $d\rho = \mu_\alpha^{1/3}d\alpha$.

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- Affine rational invariants: $J_a = \frac{(\mu_\alpha)^2}{\mu^3}$, $\quad K_a = \frac{\mu_{\alpha\alpha}}{3\,\mu^2}$.
- Projective rational invariants: $J_p = \eta^3$, $\quad K_p = \eta_\rho$.

# Differential signature for planar curves:

- Curves for which $J_G$ or $K_G$ are undefined are called $G$-exceptional.
- Definition: the $G$-signature of a non-exceptional curve $\gamma : I_\gamma \to \mathbb{R}^2$ is a planar parametric curve $\mathcal{S}_\gamma = \{ \left( J_G|_\gamma(t), K_G|_\gamma(t) \right) \mid t \in I_\gamma \}$.
- Theorem: non-exceptional curves $\alpha$ and $\beta$ are $G$-equivalent

$\Downarrow$ $\quad$ $\Uparrow$ under certain conditions

$$\mathcal{S}_\alpha = \mathcal{S}_\beta$$

# Algorithm for solving projection problems for curves:

- INPUT: a planar curve $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, and a spatial curve $\Gamma(t) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in \mathbb{R}$, with rational parameterizations.
- OUTPUT: YES or NO answer to the question "Is necessary condition for existence of finite or affine projection $[P]$ such that $[C_\gamma] = [P][C_\Gamma]$ is satisfied?".

# Outline for central projections:

1. if $\gamma$ is $\mathcal{PGL}(3)$-exceptional (straight line or conic) then follow a special procedure, else
2. for arbitrary real $c_1, c_2, c_3$ define a curve $\epsilon_{c_1,c_2,c_3}(s) = \left(\frac{z_1(s)+c_1}{z_3(s)+c_3}, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$;
3. evaluate $\mathcal{PGL}(3)$-invariants $J_p = \eta^3$, $K_p = \eta_\rho$ on $\gamma(t)$ – obtain two rational functions of $t$, $J_p|_\gamma(t)$ and $K_p|_\gamma(t)$;
4. evaluate the same $\mathcal{PGL}(3)$-invariants on $\epsilon_{c_1,c_2,c_3}(s)$ – obtain two rational functions $J_p|_\epsilon(c_1, c_2, c_3, s)$ and $K_p|_\epsilon(c_1, c_2, c_3, s)$ of $c_1, c_2, c_3$ and $s$;
5. if $\exists$ $c_1$, $c_2$, $c_3 \in \mathbb{R}$ s. t. the signatures $\mathcal{S}_\gamma = \{(J_p|_\gamma(t), K_p|_\gamma(t)) \mid t \in \mathbb{R}\}$ and $\mathcal{S}_\epsilon = \{(J_p|_\epsilon(s), K_p|_\epsilon(s)) \mid s \in \mathbb{R}\}$ coincide, then OUTPUT: $\mathrm{YES}$, else OUTPUT: $\mathrm{NO}$.

# Computational challenges:

- Elimination of variables.
- Polynomial solving over real numbers.

# Previous works:

- Projection problem for curves and surfaces for finite cameras by Feldmar, Ayache, and Betting, (1995)
  - internal camera parameters are known: *central projections* with 6 degrees of freedom;
  - additional assumption on the image curves are made.
- Projection problem for finite ordered sets of points for *parallel projections* by Arnold, Stiller, and Sturtz (2006, 2007)
  - define an algebraic variety that characterizes pairs of sets related by an affine projection;
  - define object-image distance.

# Advantages of current approach:

Universality of the framework

- applies to various types of projections and various objects.
- generalizes to higher dimensions.

# Central projections for finite sets of points:

**Projection criterion**:
A given *ordered* set $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^m)$ of $m$ points in $\mathbb{R}^3$ with coordinates $\mathbf{z}^i = (z_1^i, z_2^i, z_3^i)$, $i = 1 \ldots m$, projects onto a given *ordered* set $X = (\mathbf{x}^1, \ldots, \mathbf{x}^m)$ of $m$ points in $\mathbb{R}^2$ with coordinates $\mathbf{x}^i = (x^i, y^i)$ if and only if there exist $c_1, c_2, c_3 \in \mathbb{R}$ and $[A] \in \mathcal{PGL}(3)$ such that

$$[x^i, y^i, 1]^{tr} = [A][z_1^i + c_1,\ z_2^i + c_2,\ z_3^i + c_3]^{tr} \text{ for } i = 1 \ldots m.$$

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Use joint algebraic invariants instead of differential invariants to solve $\mathcal{PGL}(3)$-equivalence problem on the plane.

# Continuous vs. discrete:

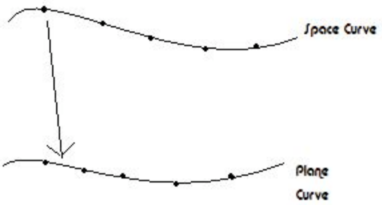


Figure: Projection problem for curves vs. projection problems for finite ordered sets of points

If $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^m)$ is a discrete sampling of a curve $\Gamma$ and $X = (\mathbf{x}^1, \ldots, \mathbf{x}^m)$ is a discrete sampling of $\gamma$, these sets might not be in a correspondence under a projection even when the curves are related by a projection.

# Further projects:

- Solutions of the projection problem for curves given by a finite sample of points.
- Object-image distance for curves.