Object-image correspondence under projections

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joint work with

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ACM Symposium on Computational Geometry 2012

Chapel Hill , June 17- 20, 2012

Projection problem:

Given: A subset $\mathcal{Z} \subset \mathbb{R}^3$ and a subset $\mathcal{X} \subset \mathbb{R}^2$.

Decide: whether there exists a projection $P\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $\mathcal{X} = P(\mathcal{Z})$

Motivation: Establishing a correspondence between objects in 3D and their images, when camera parameters and position are unknown.

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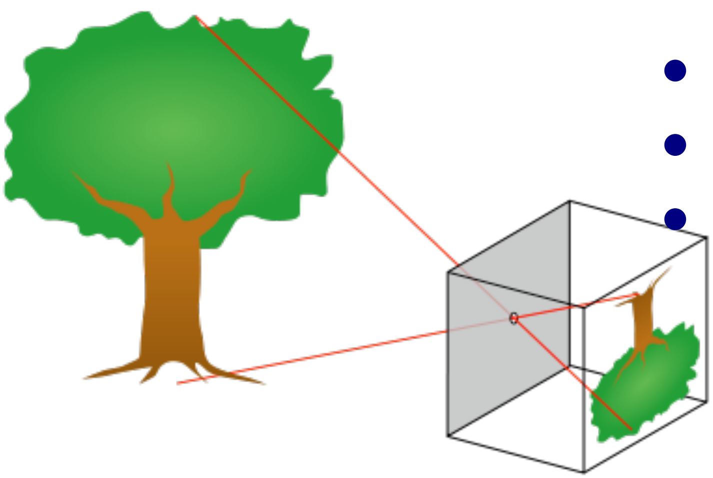
11 **degrees of freedom:**



- location of the center (3 parameters);
- position of the image plane (3 parameters);
- coordinates (not necessarily orthogonal) on the image plane (5 parameters).

Projection problem:

Given: A rational algebraic curve $\mathcal{Z} \subset \mathbb{R}^3$ and a rational algebraic curve $\mathcal{X} \subset \mathbb{R}^2$.

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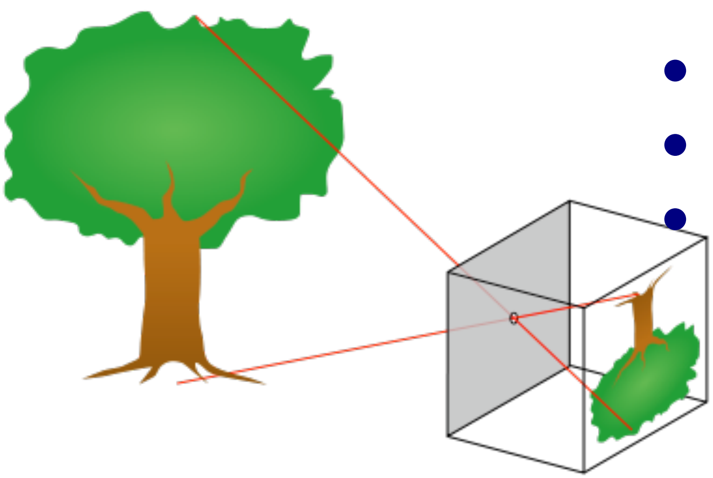

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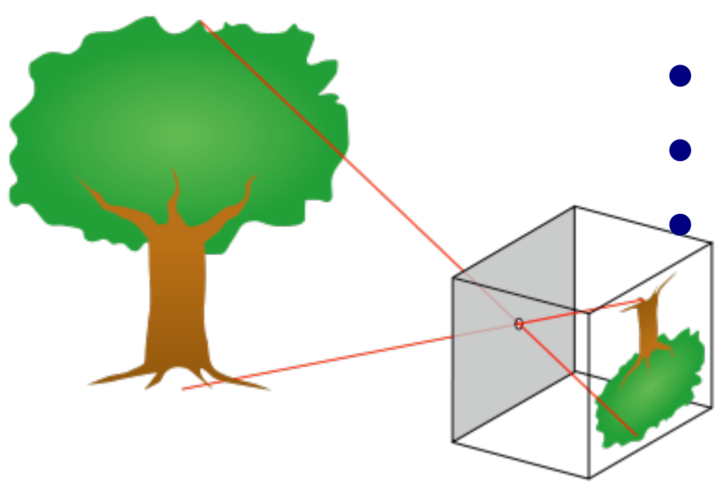

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Decide: whether there exists a projection $P\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $\mathcal{X} = \overline{P(\mathcal{Z})}$

Example: Consider the projection of $\mathcal{Z}$ parametrized by

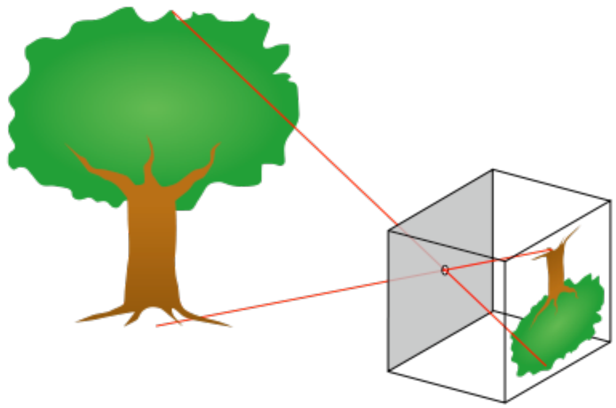
$$z_1(s) = s^3,\ z_2(s) = s^5,\ z_3(s) = s$$

from $(0,0,0)$ to the plane $z_3 = 1$: $\boxed{x = \frac{z_1}{z_3} \text{ and } y = \frac{z_2}{z_3}.}$

Then $P(Z) = \left(\frac{s^3}{s}, \frac{s^5}{s}\right) = (s^2, s^4)$ occupies half of the parabola $\mathcal{X}$ parametrized $x = t, y = t^2$ with $x > 0$.

We still say that $\mathcal{Z}$ projects to $\mathcal{X}$.

Projections:



$$P : \mathbb{R}^3 \to \mathbb{R}^2$$

$$x = \frac{p_{11}\,z_1 + p_{12}\,z_2 + p_{13}\,z_3 + p_{14}}{p_{31}\,z_1 + p_{32}\,z_2 + p_{33}\,z_3 + p_{34}},$$
$$y = \frac{p_{21}\,z_1 + p_{22}\,z_2 + p_{23}\,z_3 + p_{24}}{p_{31}\,z_1 + p_{32}\,z_2 + p_{33}\,z_3 + p_{34}}. \quad (1)$$

(1) describes

- central projection if $\det(p_{ij})_{i=1,2,3}^{j=1,2,3} \neq 0$
  (12 parameters but 11 degrees of freedom, because multiplication of all p_{ij} by the same non-zero constant gives the same projection.).
- parallel projection if denominator is a non-zero constant
  (8 parameters/degrees of freedom).

- In the paper, we consider central and parallel projections.
- In the talk, we consider central projections only.

Main idea of the algorithm

To use the relation between the projection problem and the group equivalence problem to eliminate all unknown projection parameters except the center of the projections.

Group-equivalence of planar curves

The projective group:

$$\mathcal{PGL}(3) \;=\; \{\text{equivalence classes of } 3\times 3 \text{ non-singular matrices up to multiplication by a non-zero constant.}\}$$

$\mathcal{PGL}(3)$ acts $\mathbb{R}^2$ by linear fractional transformation:

$$\bar{x} \;=\; \frac{a_{11}\,x + a_{12}\,y + a_{13}}{a_{31}\,x + p_{32}\,y + a_{33}},$$
$$\bar{y} \;=\; \frac{a_{21}\,x + a_{22}\,y + a_{23}}{a_{31}\,x + a_{32}\,y + a_{33}}.$$

Definition: We say that $\mathcal{X}_1 \subset \mathbb{R}^2$ is $\mathcal{PGL}(3)$-equivalent to $\mathcal{X}_2 \subset \mathbb{R}^2$

if $\exists A \in \mathcal{PGL}(3)$ such that $\mathcal{X}_2 = \overline{A(\mathcal{X}_1)}$

Notation: $\mathcal{X}_1 \cong \mathcal{X}_2$.

Projection criterion for algebraic curves

A curve $\mathcal{Z} \subset \mathbb{R}^3$, parametrized by $z_1(s)$, $z_2(s)$, $z_3(s)$ projects to a curve $\mathcal{X} \subset \mathbb{R}^2$ by a central projection if and only if $\exists\ c_1, c_2, c_3 \in \mathbb{R}$ such that $\mathcal{X}$ is $\mathcal{PGL}(3)$-equivalent to a planar curve parametrized by

$$\epsilon_c = \left(\frac{z_1(s) + c_1}{z_3(s) + c_3}, \quad \frac{z_2(s) + c_2}{z_3(s) + c_3} \right)$$

Remark: the projection center is $(-c_1, -c_2, -c_3)$.

Group-equivalence problem for planar curves

Problem: Let a group G act on $\mathbb{R}^2$. Given two planar algebraic curves $\mathcal{X}_1$, $\mathcal{X}_2$, decide if there exists $A \in G$ such that $\mathcal{X}_1 = \overline{A(\mathcal{X}_2)}$.

Proposed solution: is based on an algebraic adaptation of a method from differential geometry that solves local equivalence problem for smooth curves.

- In the paper, we present solution for general G.
- In the talk, $G = \mathcal{PGL}(3)$.

Rational differential invariants and signatures.

Let $\mathcal{X}$ be a rational algebraic curve with a parameterization $(x(t), y(t))$.

Classical curvatures and arclengths:

$SE(2)$: $\kappa = \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}, \quad ds = \sqrt{\dot{x}^2+\dot{y}^2}\,dt \;\Rightarrow \kappa_s = \frac{d\kappa}{ds},\ \kappa_{ss}, \ldots$

$\mathcal{SA}(2)$: $\mu = \frac{3\,\kappa\,(\kappa_{ss}+3\,\kappa^3)-5\,\kappa_s^2}{9\,\kappa^{8/3}}, \quad d\alpha = \kappa^{1/3}ds \;\Rightarrow\; \mu_\alpha = \frac{d\mu}{d\alpha},\ \mu_{\alpha\alpha}, \ldots$

$\mathcal{PGL}(3)$: $\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha-7\,\mu_{\alpha\alpha}^2-9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}, \quad d\rho = \mu_\alpha^{1/3}d\alpha \;\Rightarrow\; \eta_\rho = \frac{d\eta}{d\rho}, \ldots.$

$\boxed{K = \eta^3 \text{ and } T = \eta_\rho \text{ are rational differential } \mathcal{PGL}(3)\text{-invariants}}$

Definition. If $\mathcal{X}$ is not a line or a conic, then its $\mathcal{PGL}(3)$-signature $\mathcal{S}|_{\mathcal{X}}$ is the planar curve with rational parametrization

$$t \longrightarrow (K(t),\, T(t))$$

Theorem. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are not lines or conics then

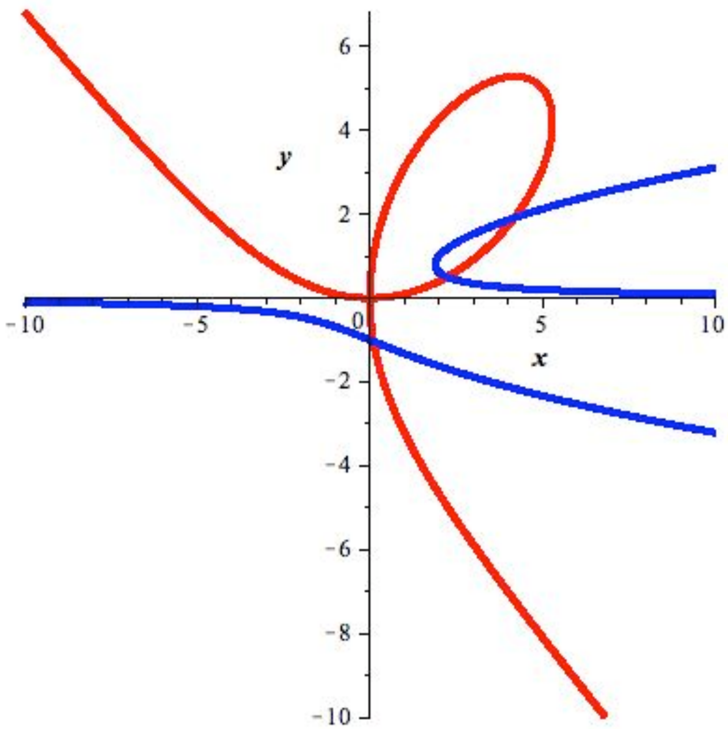
$$\boxed{\mathcal{X}_1 \cong \mathcal{X}_2 \quad \Longleftrightarrow \quad \mathcal{S}_{\mathcal{X}_1} = \mathcal{S}_{\mathcal{X}_2}.}$$

Examples of solving $\mathcal{PGL}(3)$-equivalence problem

Is $\alpha(t) = \left(\frac{10\,t}{t^3+1}, \frac{10\,t^2}{t^3+1}\right)$ implicitly defined by $x^3 + y^3 - 10\,x\,y = 0$

$\mathcal{PGL}(3)$-equivalent to

$\beta(s) = \left(\frac{s^3+3\,s^2+3\,s+2}{s+1},\ s+1\right)$ implicitly defined by $y^3 - x\,y + 1 = 0$?



- The signature $\mathcal{S}_\alpha$ for $\alpha(t) = \left(\frac{10\,t}{t^3+1}, \frac{10\,t^2}{t^3+1}\right)$ is a parametric curve

$$K|_\alpha(t) = -\frac{9261}{50}\,\frac{(t^6-t^3+1)^3\,t^3}{(t^3-1)^8},\ T|_\alpha(t) = -\frac{21}{10}\,\frac{(t^3+1)^4}{(t^3-1)^4}.$$

- The signature $\mathcal{S}_\beta$ for $\beta(s) = \left(\frac{s^3+3\,s^2+3\,s+2}{s+1},\ s+1\right)$ is a parametric curve

$$\begin{aligned} K|_\beta(s) &= -\frac{9261}{50}\,\frac{1}{(s^2+3\,s+3)^8\,s^8} \\ &\quad (s^9+9\,s^8+36\,s^7+83\,s^6+120\,s^5+111\,s^4+65\,s^3+24\,s^2+6\,s+1) \\ &\quad (s^6+6\,s^5+15\,s^4+19\,s^3+12\,s^2+3\,s+1)^2 \\ T|_\beta(s) &= -\frac{21}{10}\,\frac{(s^3+3\,s^2+3\,s+2)^4}{(s^2+3\,s+3)^4\,s^4}. \end{aligned}$$

- Is it true that $\mathcal{S}_\alpha = \mathcal{S}_\beta$ and hence α and β are $\mathcal{PGL}(3)$-equivalent?
  - $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ have the same implicit equation:

$$\begin{aligned} 0 &= 62523502209 + 39697461720\,T - 6401203200\,K + 5250987000\,T^2 \\ &- 2032128000\,K\,T + 163840000\,K^2 + 259308000\,T^3 + 53760000\,K\,T^2 \\ &+ 4410000\,T^4 \end{aligned}$$

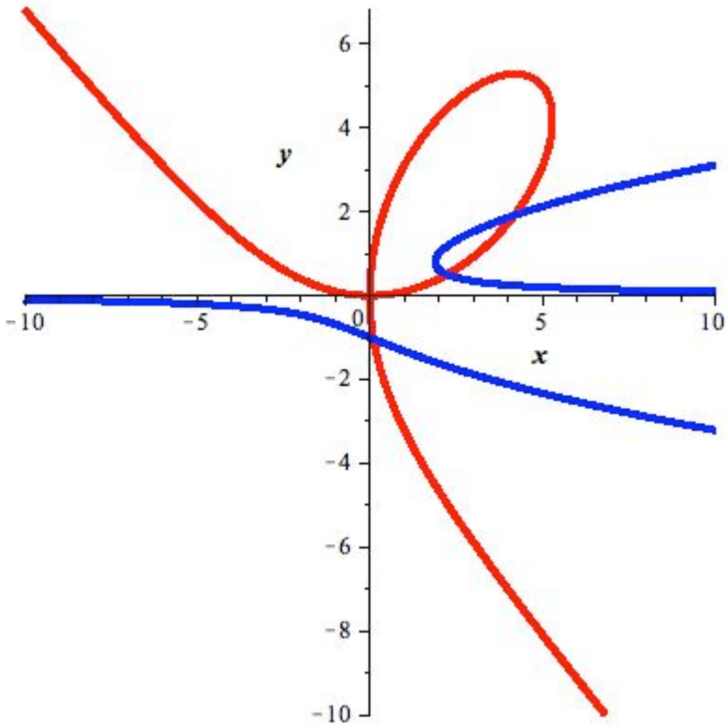
Over $\mathbb{C}$ it is a sufficient condition, but not over $\mathbb{R}$.

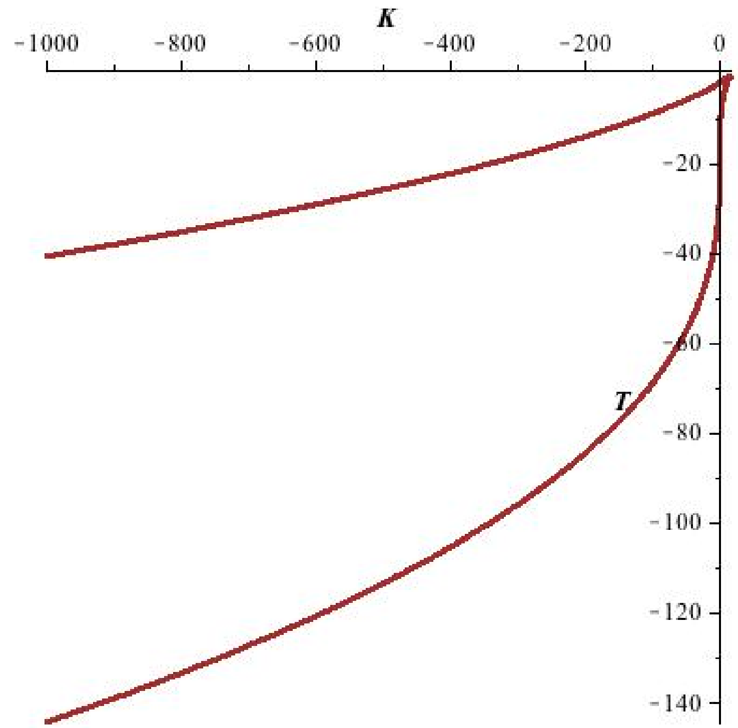
We can look for a real reparameterization by solving $T|_\alpha(t) = T|_\beta(s)$ for t in terms of s:

$t = s + 1$ indeed works. Yes!!!

The $\mathcal{PGL}(3)$ transformation that brings α to β is

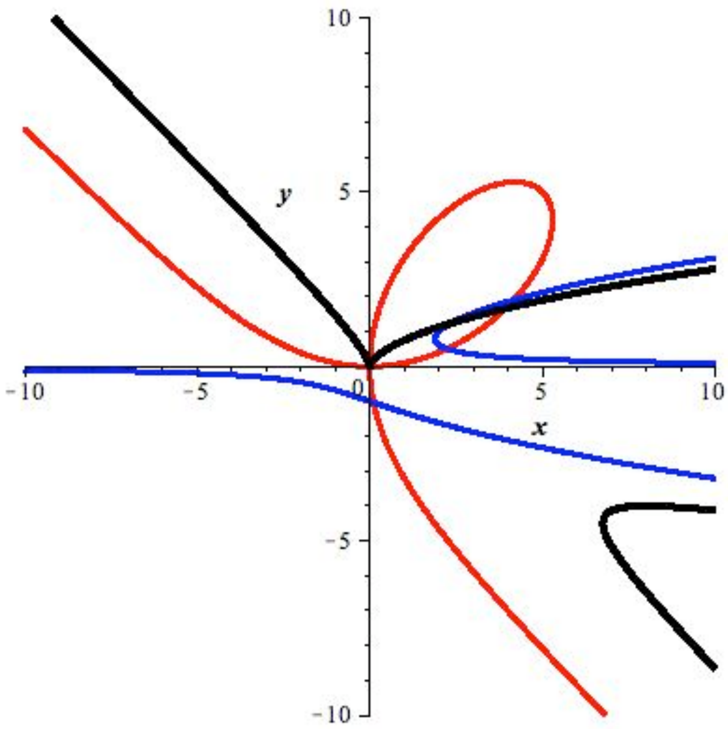
$$x \longrightarrow \frac{10\,y}{x}, \quad y \longrightarrow \frac{10}{x}.$$





Curves $\mathcal{X}_1$ and $\mathcal{X}_2$ have the same signature

Is $\gamma(w) = \left(\frac{w^3}{w+1}, \frac{w^2}{w+1}\right)$ implicitly defined by $y^3 - x^2 + x\,y^2 = 0$ $\mathcal{PGL}(3)$-equivalent to α and β?



No! because its signature is different:

$$K|_\gamma(w) = \frac{250047}{12800} \text{ and } T|_\gamma(w) = 0$$

and so $\mathcal{S}_\gamma = \left(\frac{250047}{12800}, 0\right)$ is a point!

Algorithm for central projections.

INPUT: Rational parameterizations $(z_1(s), z_2(s), z_3(s)) \in \mathbb{Q}(s)^3$ and $(x(t), y(t)) \in \mathbb{Q}(t)^2$ of algebraic curves $\mathcal{Z} \subset \mathbb{R}^3$ and $\mathcal{X} \subset \mathbb{R}^2$, where $\mathcal{Z}$ is not a line.

OUTPUT: The truth of the statement:

$$\exists \text{ central projection } P \text{ such that } \mathcal{X} = \overline{P(\mathcal{Z})}.$$

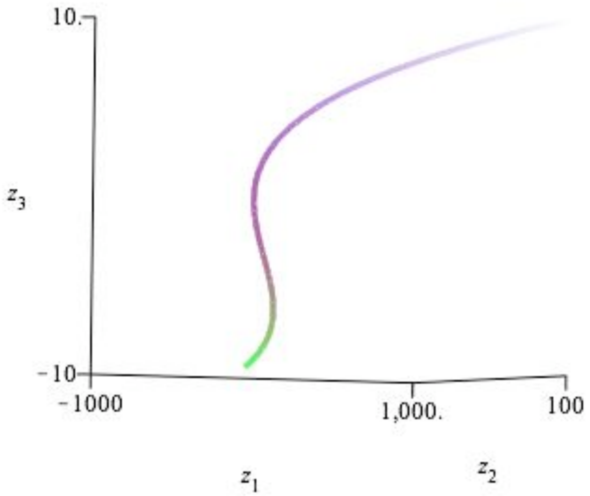
NON-RIGOROUS OUTLINE:

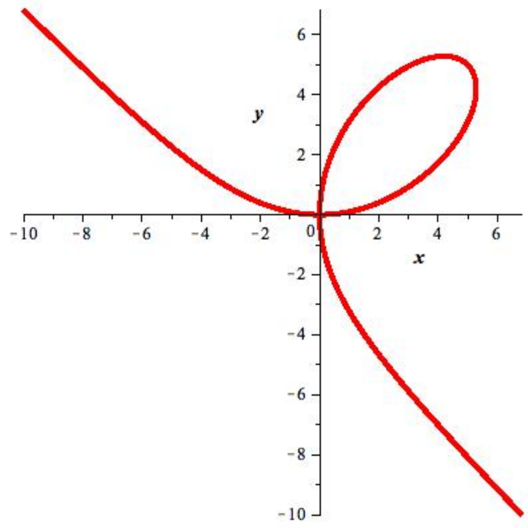
1. if $\mathcal{X}$ is a line then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\mathcal{Z}$ is coplanar.
2. $\epsilon_c := \left(\frac{z_1(s)+c_1}{z_3(s)+c_3}, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$ is a family of parametric curves.
3. if $\mathcal{X}$ is a conic then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c = (c_1, c_2, c_3)$, such that $\epsilon_c(s)$ parametrizes a conic.
4. if $\mathcal{X}$ is neither a line or a conic then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c$ such that the signature of the curve parametrized by $\epsilon_c(s)$ is contained in the signature of $\mathcal{X}$.

Example: central projections of the twisted cubic

Can the twisted cubic $\mathcal{Z}$ parametrized by

$$\Gamma(s) = \left(s^3,\, s^2,\, s\right)\ ,\ s \in \mathbb{R}$$





be projected to a curve $\mathcal{X}_1$ parametrized by $\alpha(t) = \left(\frac{10\,t}{t^3+1},\, \frac{10\,t^2}{t^3+1}\right)$ with an implicit equation $x^3 + y^3 - 10\,y\,x = 0$?

- The signature of $\mathcal{X}_1$ is parametrized by invariants:

$$K|_\alpha(t) = -\frac{9261}{50}\frac{(t^6 - t^3 + 1)^3\, t^3}{(t^3 - 1)^8},\; T|_\alpha(t) = -\frac{21}{10}\frac{(t^3 + 1)^4}{(t^3 - 1)^4}.$$

- Compute invariants $K|_\epsilon(c, s)$ and $T|_\epsilon(c, s)$ for the curve $\epsilon_c(s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$ with indetreminant values of c.

- Does there exist c such that $(K|_\epsilon(c, s),\, T|_\epsilon(c, s))$ parametrize the same signature as $(K|_\alpha(t),\, T|_\alpha(t))$?

- This is indeed true for c=(1,0,0).

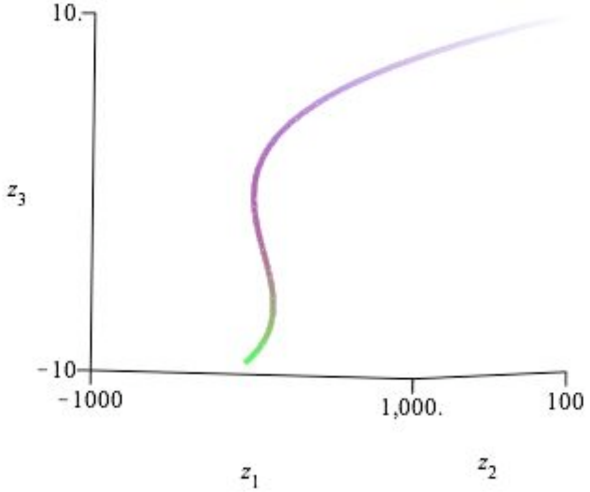
- Yes!! The twisted cubic can be projected to $x^3 + y^3 - 10\, y\, x = 0$.

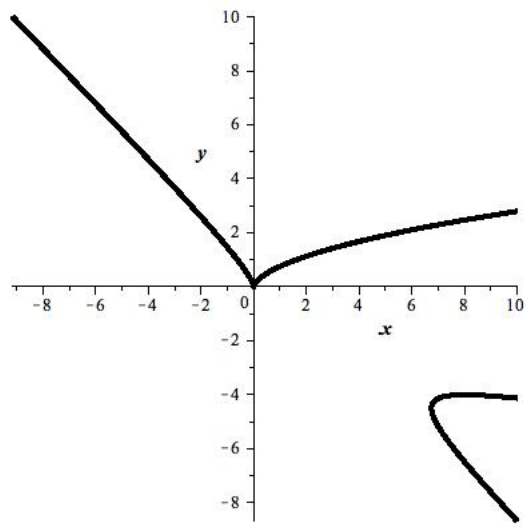
- A possible projection is $x = \frac{10\, z_3}{z_1+1}$, $y = \frac{10\, z_2}{z_1+1}$.

It follows that the twisted cubic can be projected to $\mathcal{X}_2$ because $\mathcal{X}_1 \cong \mathcal{X}_2$.

Can the twisted cubic $\mathcal{Z}$ parametrized by

$$\Gamma(s) = \left(s^3, s^2, s\right), \; s \in \mathbb{R}$$





be projected to a curve $\mathcal{X}_3$ parametrized by $\gamma(t) = \left(\frac{t^3}{t+1}, \frac{t^2}{t+1}\right)$ with an implicit equation $y^3 + y^2 x - x^2 = 0$?

- The signature of $\mathcal{X}_3$ degenerates to a point.

$$K|_\gamma(t) = \frac{250047}{12800} \text{ and } T|_\gamma(t) = 0, \quad \forall t \in \mathbb{R}.$$

- We need to determine if there exists c such that a curve parametrized by $\epsilon_c(s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$ has the same constant invariants as $\mathcal{X}_3$.

- This is indeed true for c=(0,0,1).

- Yes!! The twisted cubic can be projected to $y^3 + y^2\,x - x^2 = 0$.

- A possible projection is $x = \frac{z_1}{z_3+1}$, $y = \frac{z_2}{z_3+1}$.

- Recall that $\mathcal{X}_3$ is <u>not</u> $\mathcal{PGL}(3)$-equivalent to $\mathcal{X}_1$ and $\mathcal{X}_2$.

Can the twisted cubic be projected to a parabola parametrized by (t, t^2)?

- Does there exists c such that a curve parametrized by

$$\epsilon_c(s) = \left(\frac{s^3 + c_1}{s + c_3}, \frac{s^2 + c_2}{s + c_3}\right)$$

  is a quadric?

- Yes!! $c_1 = c_2 = c_3 = 0$

Can the twisted cubic be projected to quintic parameterized by (t, t^5)?

- The signature of the quintic degenerates to a point:

$$K(t) = \frac{1029}{128} \text{ and } T(t) = 0\,, \quad \forall t.$$

- Does there exists c such that

$$K|_\epsilon(c, s) = \frac{1029}{128} \text{ and } T|_\epsilon(c, s) = 0,\ \forall s \in \mathbb{R}?$$

- NO!! Substitution of several values of s gives an inconsistent system on c.

Previous works

Finite lists of points

- Hartley and Zisserman (2004) set up a system of conditions on the projection parameters and then check whether or not this system has a solution.

- Arnold, Stiller, and Sturtz (2006, 2007) define an algebraic variety that characterizes pairs of lists related by a parallel projection.

Curves and surfaces

- Feldmar, Ayache, and Betting (1995) set up a system of conditions on the projection parameters with known internal parameters (central projections with 6 unknown parameters vs 12 considered here).

Advantage of our approach

- We need to eliminate 3 projection parameters instead of 12. In general, the less parameters to eliminate – the better (although other factors may be important).
- The same approach can be used in the case of parallel projections.
- Our approach can be used for finite lists of points (with signatures based on a separating set of algebraic invariants)

Implementation: The projection problem can be considered over $\mathbb{C}$ and the proposed method is easier to implement over $\mathbb{C}$.

Maple Code

`www.math.ncsu.edu/~iakogan/symbolic/projections.html`

Can we use the same method to solve the projection problem for non-rational curves?

In principle, yes, but

one has to be careful when describing a family of planar curves

$$\tilde{\mathcal{Z}}_c = \overline{\left\{ \left(\frac{z_1 + c_1}{z_3 + c_3}, \frac{z_2 + c_2}{z_3 + c_3} \right) \middle| (z_1, z_2, z_3) \in \mathcal{Z} \right\}}$$

by an implicit equation.

Assume $\mathcal{Z}$ is given by implicit equations $g_1(z_1, z_2, z_3) = 0$, $g_2(z_1, z_2, z_3) = 0$. For fixed c_1, c_2, c_3 we need to eliminate z_1, z_2, z_3 from the equations

$$\begin{aligned} 0 &= g_1(z_1, z_2, z_3) \\ 0 &= g_2(z_1, z_2, z_3) \\ x &= \frac{z_1 + c_1}{z_3 + c_3} \\ y &= \frac{z_2 + c_2}{z_3 + c_3} \end{aligned}$$

Unfortunately, in general, elimination does not commute with specialization of the parameters c_1, c_2, c_3 .

Example: the twisted cubic is implicitly defined by equations

$$z_1 - z_2\, z_3 = 0, \quad z_2 - z_3^2 = 0$$

If c is such that $c_2 \neq -c_3^2$ and $c_1 \neq c_3^3$, elimination of z_1, z_2, z_3 from the equations

$$z_1 - z_2\, z_3 = 0, \qquad x = \frac{z_1 + c_1}{z_3 + c_3},$$
$$z_2 - z_3^2 = 0, \qquad y = \frac{z_2 + c_2}{z_3 + c_3}.$$

leads to

$$\begin{aligned} 0 \;=\; & (-c_3^2 - c_2)\, x^2 + (c_3^2 + c_2)\, y^2\, x + (c_1 + c_3\, c_2)\, x\, y + \\ & (2\, c_1\, c_3 - 2\, c_2^2)\, x + (c_3^3 - c_1)\, y^3 + (-3\, c_1\, c_3 - 3\, c_3^2\, c_2)\, y^2 + \\ & (3\, c_2^2\, c_3 + 3\, c_1\, c_2)\, y - c_1^2 - c_2^3 \end{aligned}$$

If $c_2 = -c_3^2$ and $c_1 = c_3^3$, the elimination leads to

$$y^2 - x + c_3\, y + c_3^2 = 0.$$

Projection criterion for list of points*:

A list $\mathbf{Z} = (\mathbf{z}^1, \ldots, \mathbf{z}^m)$ of m points with coordinates $\mathbf{z}^i = (z_1^r, z_2^r, z_3^r)$, $r = 1 \ldots m$, projects onto a list $X = (\mathbf{x}^1, \ldots, \mathbf{x}^m)$ of m points in $\mathbb{R}^2$ with coordinates $\mathbf{x}^r = (x^r, y^r)$ by a finite projection if and only if there exist $c_1, c_2, c_3 \in \mathbb{R}$ and $[A] \in \mathcal{PGL}(3)$, such that

$$[x^r, y^r, 1]^T = [A][z_1^r + c_1,\ z_2^r + c_2,\ z_3^r + c_3]^T \text{ for } r = 1 \ldots m.$$

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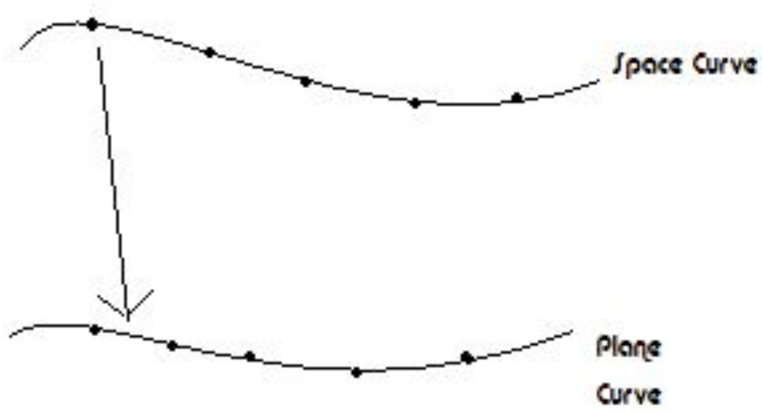
*separating sets of algebraic invariants can be used to solve group-equivalence problems for sets of points

Continuous vs. discrete:

Projection problem for curves vs. projection problems for finite lists of points.



If $\mathbf{Z} = (\mathbf{z}^1, \ldots, \mathbf{z}^m)$ is a discrete sampling of a curve $\mathcal{Z}$ and $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^m)$ is a discrete sampling of $\mathcal{X}$, these sets might not be in a correspondence under a projection even when the curves $\mathcal{Z}$ and $\mathcal{X}$ are related by a projection.

Thank you !!! *

*Additional slides follow

Differentially separating set of rational $\mathcal{PGL}(3)$-invariants:

$$\boxed{\Delta_2 = 9\,y^{(5)}\,[y^{(2)}]^2 - 45\,y^{(4)}\,y^{(3)}\,y^{(2)} + 40\,[y^{(3)}]^3.}$$

$$
\begin{aligned}
K_{\mathcal{P}} &= \frac{729}{8\,(\Delta_2)^8}\Bigl(18\,y^{(7)}\,[y^{(2)}]^4\,\Delta_2 - 189\,[y^{(6)}]^2\,[y^{(2)}]^6 \\
&+ 126\,y^{(6)}\,[y^{(2)}]^4\,(9\,y^{(5)}\,y^{(3)}\,y^{(2)} + 15\,[y^{(4)}]^2\,y^{(2)} - 25\,y^{(4)}\,[y^{(3)}]^2) \\
&- 189\,[y^{(5)}]^2\,[y^{(2)}]^4\,(4\,[y^{(3)}]^2 + 15\,y^{(2)}\,y^{(4)}) \\
&+ 210\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2\,(63\,[y^{(4)}]^2\,[y^{(2)}]^2 - 60\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} + 32\,[y^{(3)}]^4) \\
&- 525\,y^{(4)}y^{(2)}\,(9\,[y^{(4)}]^3\,[y^{(2)}]^3 + 15\,[y^{(4)}]^2\,[y^{(3)}]^2\,[y^{(2)}]^2 - 60\,y^{(4)}\,[y^{(3)}]^4\,y^{(2)} + 64\,[y^{(3} \\
&+ 11200\,[y^{(3)}]^8\Bigr)^3;
\end{aligned}
$$

$$
\begin{aligned}
T_{\mathcal{P}} &= \frac{243\,[y^{(2)}]^4}{2\,(\Delta_2)^4}\Bigl(2\,y^{(8)}\,y^{(2)}\,(\Delta_2)^2 \\
&- 8\,y^{(7)}\,\Delta_2\,(9\,y^{(6)}\,[y^{(2)}]^3 - 36\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2 - 45\,[y^{(4)}]^2\,[y^{(2)}]^2 + 120\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} \\
&+ 504\,[y^{(6)}]^3\,[y^{(2)}]^5 - 504\,[y^{(6)}]^2\,[y^{(2)}]^3\,(9\,y^{(5)}\,y^{(3)}\,y^{(2)} + 15\,[y^{(4)}]^2\,y^{(2)} - 25\,y^{(4)}\,[y^{(3)} \\
&+ 28\,y^{(6)}\,\bigl(432\,[y^{(5)}]^2\,[y^{(3)}]^2\,[y^{(2)}]^3 + 243\,[y^{(5)}]^2\,y^{(4)}\,[y^{(2)}]^4 - 1800\,y^{(5)}\,y^{(4)}\,[y^{(3)}]^3\,[y^{(2} \\
&- 240y^{(5)}\,[y^{(3)}]^5\,y^{(2)} + 540y^{(5)}\,[y^{(4)}]^2\,[y^{(3)}]\,[y^{(2)}]^3 + 6600\,[y^{(4)}]^2\,[y^{(3)}]^4\,y^{(2)} - 2000y \\
&- 5175\,[y^{(4)}]^3\,[y^{(3)}]^2\,[y^{(2)}]^2 + 1350\,[y^{(4)}]^4\,[y^{(2)}]^3\bigr) - 2835\,[y^{(5)}]^4\,[y^{(2)}]^4 \\
&+ 252\,[y^{(5)}]^3y^{(3)}\,[y^{(2)}]^2\,(9y^{(4)}\,y^{(2)} - 136\,[y^{(3)}]^2) - 35840\,[y^{(5)}]^2\,[y^{(3)}]^6 \\
&- 630\,[y^{(5)}]^2\,[y^{(4)}]\,[y^{(2)}]\,(69\,[y^{(4)}]^2\,[y^{(2)}]^2 - 160\,[y^{(3)}]^4 - 153\,y^{(4)}\,[y^{(3)}]^2\,[y^{(2)}]) \\
&+ 2100\,y^{(5)}\,[y^{(4)}]^2\,y^{(3)}\,(72\,[y^{(3)}]^4 + 63\,[y^{(4)}]^2\,[y^{(2)}]^2 - 193\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)}) \\
&- 7875\,[y^{(4)}]^4\,(8\,[y^{(4)}]^2\,[y^{(2)}]^2 - 22y^{(4)}\,[y^{(3)}]^2\,[y^{(2)}] + 9\,[y^{(3)}]^4)\Bigr).
\end{aligned}
$$

The restriction of $K_{\mathcal{P}}|_{\mathcal{X}}$ and $T_{\mathcal{P}}|_{\mathcal{X}}$ to a planar curve $\mathcal{X}$ with rational parameterization $(x(t), y(t))$ is computed by substitution

$$y^{(1)} = \frac{\dot{y}}{\dot{x}} \quad , \ldots , \quad y^{(k)} = \frac{\dot{y^{(k-1)}}}{\dot{x}}, \tag{3}$$

into the formulas for invariants.

- $y^{(1)}, \ldots, y^{(k)}$ are rational functions of t unless $\mathcal{X}$ is a vertical line.
- Invariants $K_{\mathcal{P}}|_{\mathcal{X}}$ and $T_{\mathcal{P}}|_{\mathcal{X}}$ are rational functions of t unless $\Delta_2|_{\mathcal{X}} \underset{\mathbb{R}(t)}{=} 0$.
- $\Delta_2|_{\mathcal{X}} \underset{\mathbb{R}(t)}{=} 0$ if and only if $\mathcal{X}$ is a line or a conic.
- When the restriction of invariants to the family of curves $\tilde{\mathcal{Z}}_c$ parametrized by $\epsilon(c, s) := \left(\frac{z_1(s)+c_1}{z_3(s)+c_3}, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$ is computed the differentiation in (3) is taken with respect to s.
- For the values c, such that $\epsilon(c, s)$ is not a line or a conic, specialization of c commutes with restriction of invariants $K_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}$ and $T_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}$.

ALGORITHM:

1. if $\begin{vmatrix} \dot{\gamma} \\ \ddot{\gamma} \end{vmatrix} \underset{\mathbb{R}(t)}{=} 0$ then if $\begin{vmatrix} \dot{\Gamma} \\ \ddot{\Gamma} \\ \dddot{\Gamma} \end{vmatrix} \underset{\mathbb{R}(s)}{=} 0$ then return TRUE else return FALSE;
2. $\epsilon := \left(\frac{z_1+c_1}{z_3+c_3}, \frac{z_2+c_2}{z_3+c_3}\right) \in \mathbb{Q}(c_1, c_2, c_3, s)^2$;
3. if $\Delta_2|_\gamma \underset{\mathbb{R}(t)}{=} 0$ then if $\exists (c_1, c_2, c_3) \in \mathbb{R}^3$
$$z_3 + c_3 \underset{\mathbb{R}(s)}{\neq} 0 \wedge \begin{vmatrix} \dot{\epsilon} \\ \ddot{\epsilon} \end{vmatrix} \underset{\mathbb{R}(s)}{\neq} 0 \wedge \Delta_2|_\epsilon \underset{\mathbb{R}(s)}{=} 0$$
then return TRUE else return FALSE.
4. return the truth of the statement:

$$\exists\, (c_1, c_2, c_3) \in \mathbb{R}^3$$
$$z_3 + c_3 \underset{\mathbb{R}(s)}{\neq} 0 \wedge \begin{vmatrix} \dot{\epsilon} \\ \ddot{\epsilon} \end{vmatrix} \underset{\mathbb{R}(s)}{\neq} 0 \wedge \Delta_2|_\epsilon \underset{\mathbb{R}(s)}{\neq} 0 \tag{4}$$
$$\wedge \forall s \in \mathbb{R}$$
$$\Delta_2|_\epsilon \underset{\mathbb{R}}{\neq} 0 \;\Rightarrow\; \exists t \in \mathbb{R}$$
$$K_{\mathcal{P}}|_\epsilon \underset{\mathbb{R}}{=} K_{\mathcal{P}}|_\gamma \wedge T_{\mathcal{P}}|_\epsilon \underset{\mathbb{R}}{=} T_{\mathcal{P}}|_\gamma.$$