

MA 408 Computer Lab Two  
**The Poincaré Disk Model of Hyperbolic Geometry**

Put your name here: \_\_\_\_\_

Score: \_\_\_\_\_

**Instructions:** For this lab you will be using the applet, NonEuclid, created by Castellanos, Austin, Darnell, & Estrada. You can either download it from Vista (Go to Labs, click on NonEuclid), or from the following web address:  
<http://www.cs.unm.edu/~joel/NonEuclid/NonEuclid.html> (Click on Download NonEuclid.Jar).

Work through this lab independently or in pairs. You will have the opportunity to finish anything you don't get done in lab for homework. Notice that some questions are marked as Homework problems. Please, save these problems until you are done with the rest of the lab, and then return to them at the end of the class or at home.

## 1 Introduction

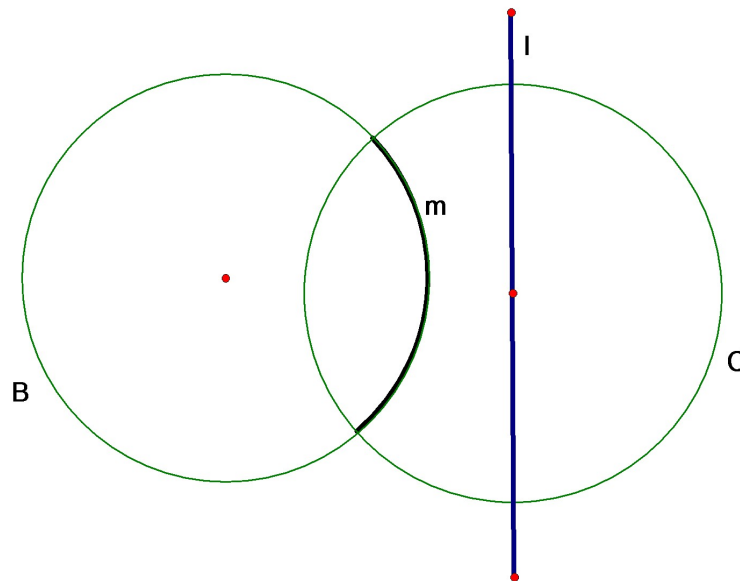


Figure 1: Lines in the Poincaré Disk Model

The Poincaré disk model is a model for hyperbolic geometry that uses Euclidean geometry.

Fix a circle,  $C$ , in the Euclidean plane. The plane in the Poincaré disk model consists of all points strictly inside  $C$ .

There are two types of lines in the Poincaré disk model illustrated on Figure 1.

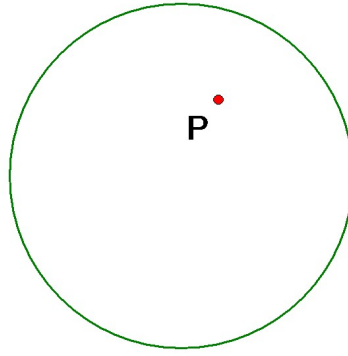


Figure 2:

- i Intersections of Euclidean lines that pass through the center of  $C$  with the interior of the disk (diameters without endpoints).
  - ii Intersections of Euclidean circles orthogonal to  $C$  with the interior of the disk.
1. Sketch several lines passing through the point  $P$  in the Poincaré disk pictured in Figure 2.
  2. Find the similarities and the differences between the Poincaré disk model with the Klein disk model in Chapter 2.
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3. Start NonEuclid. You will first need to go to File  $\rightarrow$  New. Use Constructions  $\rightarrow$  Draw Line and click on two points in the Poincaré Disk to create a line. Create two lines through the center of the disk. Then create two lines that do not pass through the center of the disk. Use Edit  $\rightarrow$  Move Point to move the lines around.

Make more experiments. Based on your observations, which of the following axioms from Chapter 2 are satisfied?

- *Incidence Axiom 1:* For every pair of distinct points  $P$  and  $Q$  there exists exactly one line  $l$  such that both  $P$  and  $Q$  lie on  $l$ .
- *Incidence Axiom 2:* For every line  $l$  there exist at least two distinct points  $P$  and  $Q$  such that both  $P$  and  $Q$  lie on  $l$ .
- *Incidence Axiom 3:* There exist three points that do not all lie on any one line.

## 2 Parallel Lines in the Poincaré Disk Model

4. State the definition of parallel lines.
  
  
  
  
  
  
  
  
  
  
5. Using the computer model, draw a line  $l$  (Constructions → Draw Line) and a point  $P$ , not on  $l$  (Constructions → Plot Point).
  - (a) Construct three lines that go through  $P$  and are parallel to  $l$ .
  - (b) How many such lines exist?
  
  
  
  
  
  
  
  - (c) Which parallel postulate does this model satisfy (Euclidean, Elliptic, or Hyperbolic found in Venema, 2002, pp. 25-26)? Explain.

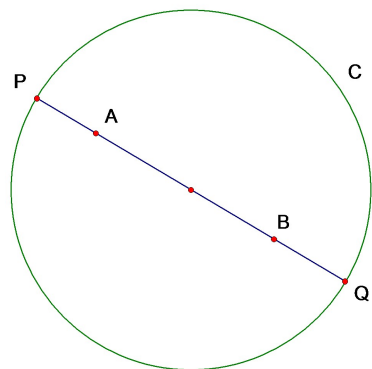


Figure 3: Points A and B lie on a Diameter

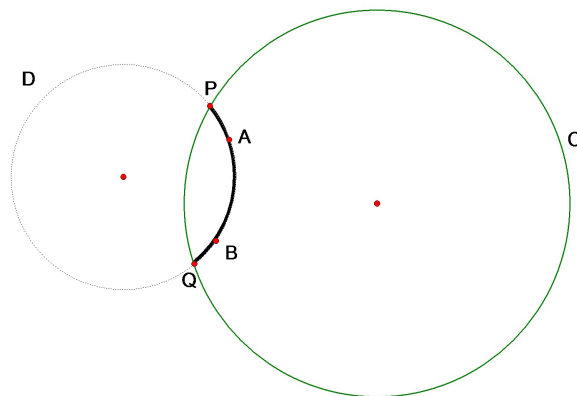


Figure 4: Points A and B lie on a Circle D Perpendicular to Circle C.

### 3 Distance in the Poincaré Disk Model

To find distance between two points in the Poincaré disk model, connect them with a Poincaré line. There are two cases (see Figure 3 and Figure 4):

- Points  $A$  and  $B$  lie on the diameter of circle  $C$  and points  $P$  and  $Q$  are endpoints of the diameter.
- Points  $A$  and  $B$  lie on circle  $D$  that is perpendicular to  $C$  and points  $P$  and  $Q$  are the points where  $D$  intersects  $C$ .

In both cases, the distance between two points  $A$  and  $B$  in the Poincaré model is defined by the formula:

$$Dist(A, B) = \left| \ln\left(\frac{AP \cdot BQ}{AQ \cdot BP}\right) \right|,$$

where  $AP, BQ, AQ,$  and  $BP$  are the usual Euclidean distances.

6. Recall the properties of logarithmic functions:

- For what values of  $x$  is  $\ln(x) > 0$ ?
- For what values of  $x$  is  $\ln(x) = 0$ ?
- For what values of  $x$  is  $\ln(x) < 0$ ?
- Simplify  $\ln(x) - \ln(y)$ .

7. Use the properties of logarithms to prove that the following three properties of distance are satisfied in the Poincaré model.

(a)  $Dist(A, B) = Dist(B, A)$ .

(b)  $Dist(A, B) \geq 0$ .

(c)  $Dist(A, B) = 0$  if and only if  $A = B$ .

Note: The triangle inequality also holds, but you do not need to prove it.

8. What happens with  $Dist(A, B)$  when  $A$  is fixed and  $B$  approaches  $Q$ ? This is  $\lim_{B \rightarrow Q} Dist(A, B)$ .

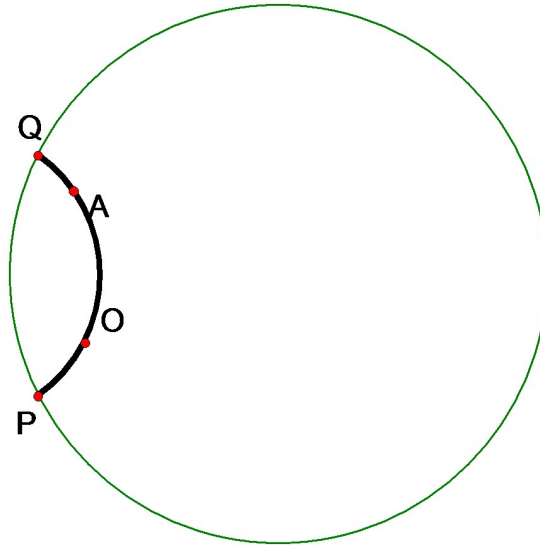


Figure 5:

9. (**h/w problem**) For each line  $l$  we fix a point  $O$  (the origin) on  $l$  and define a coordinate function  $f$  from  $l$  to the real numbers by the following formula (see Figure 5):

$$f(A) = \ln \left( \frac{AP \cdot OQ}{AQ \cdot OP} \right).$$

Compare it with the distance formula. Note that the coordinate of a point  $A$  is the distance from the origin  $O$  to the point  $A$  taken with  $+$  or  $-$  depending on its position relative to  $O$ . The following exercises indicate that  $f$  is indeed a one-to-one correspondence and it is compatible with the definition of the distance.

- (a) Compute  $f(O)$ .

(b) Is  $f(A)$  positive or negative when  $A$  is between  $Q$  and  $O$  (i.e.  $Q * A * O$  as in Figure 5)? Why?  
(Hint: Use that  $AP > OP$  and  $OQ > AQ$ , as well as properties of  $\ln$ -function).

(c) Is  $f(A)$  positive or negative when  $A$  is between  $P$  and  $O$  (i.e.  $P * A * O$ )? Why?  
Hint: Draw the corresponding picture. Note that  $OP > AP$  and  $AQ > OQ$ .

(d) What happens to the values of  $f$  when  $A \rightarrow Q$ ? (Note: This is  $\lim_{A \rightarrow Q} f$ .)

(e) What happens to the values of  $f$  when  $A \rightarrow P$ ? (Note: This is  $\lim_{A \rightarrow P} f$ .)

(f) Using the properties of logarithms, show that  $|f(A) - f(B)| = \text{Dist}(A, B)$ .

10. Using the computer model, use Constructions  $\rightarrow$  Draw Segment of Specific Length. Choose a length and enter it into the box on the left side of the screen. Then click on the Poincaré disk to create your first endpoint,  $A$ , of the segment. Click on the disk in another place to define an unmarked ray with endpoint  $A$ . It is on this ray, where the segment of the specified length will lie. Create at least 6 segments in this way. Remember that all segments that you create have the same Poincaré length.

(a) Do all of the segments you created appear to have the same Euclidean length?

(b) What do you notice as the segments get closer to the boundary of the Poincaré disk model? (You may use Edit  $\rightarrow$  Move Points to move your segments.)

(c) Try to create a segment of length 15. The computer program, will ask to choose the length between 0 and 10. Is it a theoretical or a computational restriction? Explain your answer.



- (d) Theoretically, in a Poincaré disk model, are there any bounds on the lengths of the segments? Explain your answer.

## 4 Measuring Angles in the Poincaré Disk Model

Given 2 rays which intersect at  $A$ , the measure of the angle between the Poincaré rays,  $AB$  and  $AC$  is the same as the measure of the Euclidean angle between the tangent rays,  $AB^*$  and  $AC^*$ .

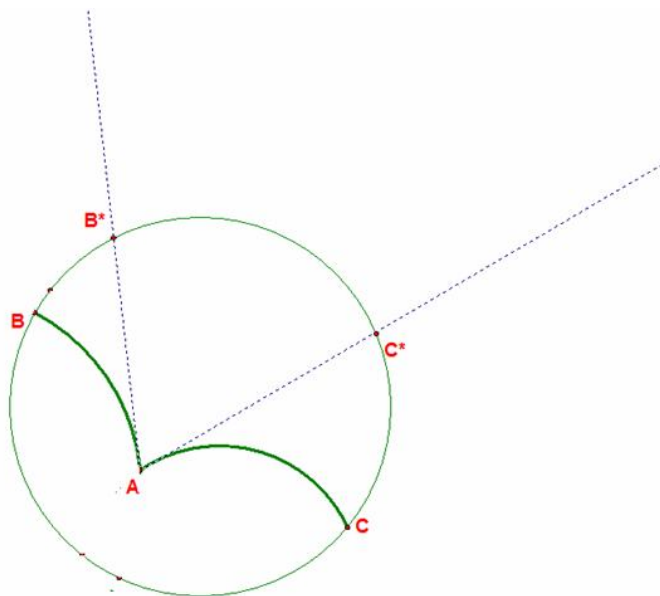


Figure 6: Measuring Angles

11. What is the sum of the measures of the angles in a triangle in Euclidean geometry?

12. Using the Poincaré disk computer model, create a triangle. Then use Measurements  $\rightarrow$  Measure Triangle to answer the following questions.

(a) Record the sum of the measures of all angles,  $\sigma(\triangle ABC)$ , in this triangle using the Poincaré model.

(b) Create two more triangles and record the sum of their measures as well.

(c) Are all answers the same?

13. **Definition:** The defect of  $\triangle ABC$  is the difference between the sum of the measures of all angles and  $180^\circ$ . Notation:  $\delta(\triangle ABC) = 180^\circ - \sigma(\triangle ABC)$ .

(a) What is the defect of each of your triangles?

(b) What is the defect of a triangle in Euclidean geometry?

14. **(h/w problem)** Prove that in Euclidean geometry the sum of the measures of all angles in a convex quadrilateral is  $360^\circ$ . (you may use the theorem about the sum of the measures of all angles in a triangle)
15. **Definition:** The defect of a quadrilateral,  $\square ABCD$ , is the difference between the sum of the measure of all angles and  $360^\circ$ . Notation:  $\delta(\square ABCD) = 360^\circ - \sigma(\square ABCD)$ .
- (a) Either use the computer model or an argument similar to the argument used to prove number 12 to make a conjecture about the sum of the measures of all angles of convex quadrilaterals in the Poincaré model.
- (b) Can you construct a rectangle in the Poincaré model? If yes, draw your result below. Mark the lengths of each side and the measure of each angle using Poincaré distance. If no, explain why. Remember a rectangle is a quadrilateral with four right angles.

## 5 Other Constructions

16. Using the computer model, use Constructions  $\rightarrow$  Draw circle to draw a circle in the Poincaré disk model. Create several circles with various centers and radii.

Create a point on one of the circles using Constructions  $\rightarrow$  Plot Point on Object. Then measure the distance from the center to this point on the circle using Measurements  $\rightarrow$  Measure Distance. Do this for several other points on the same circle.

- (a) Is the center of the circle equidistant from every point on the boundary? (Remember to use Poincaré distance.) Is it true for every circle? Why or why not?

- (b) Does the center appear equidistant in the Euclidean metric? Explain.