MA 408, Computer Lab Three

Saccheri and Lambert Quadrilaterals in Hyperbolic Geometry

Name: ________________________

Score: ________________________

Instructions: For this lab you will be using the applet, NonEuclid, developed by Castellanos, Austin, Darnell, & Estrada. You can either download it from Vista (Go to Labs, click on NonEuclid), or from the following web address:
http://www.cs.unm.edu/∼joel/NonEuclid/NonEuclid.html. (Click on Download NonEuclid.Jar).

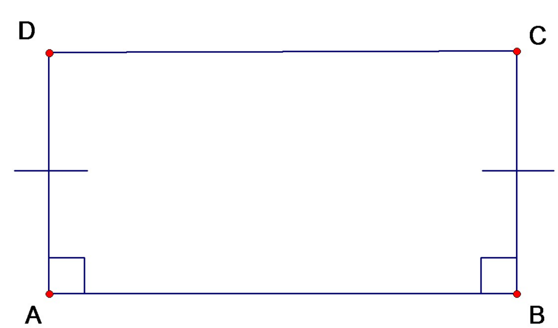
Work through this lab independently or in pairs. You will have the opportunity to finish anything you don't get done in lab at home. Please, write the solutions of homework problems on a separate sheets of paper and staple them to the lab.

1 Introduction

In the previous lab you observed that it is impossible to construct a rectangle in the Poincaré disk model of hyperbolic geometry. In this lab you will study two types of quadrilaterals that exist in the hyperbolic geometry and have many properties of a rectangle. A *Saccheri quadrilateral* has two right angles adjacent to one of the sides, called the base. Two sides that are perpendicular to the base are of equal length. A *Lambert quadrilateral* is a quadrilateral with three right angles. In the Euclidean geometry a Saccheri or a Lambert quadrilateral has to be a rectangle, but the hyperbolic world is different ...

2 Saccheri Quadrilaterals

Definition: A *Saccheri quadrilateral* is a quadrilateral $\square ABCD$ such that $\angle ABC$ and $\angle DAB$ are right angles and $\overline{AD} \cong \overline{BC}$. The segment $\overline{AB}$ is called the *base* of the Saccheri quadrilateral and the segment $\overline{CD}$ is called the *summit*. The two right angles are called the *base angles* of the Saccheri quadrilateral, and the angles $\angle CDA$ and $\angle BCD$ are called the *summit angles* of the Saccheri quadrilateral. A Saccheri quadrilateral looks like this:



Construction: We are going to construct a Saccheri quadrilateral using the Poincaré disk applet.

1. Use Constructions → Draw Line Segment to construct a horizontal segment, $\overline{AB}$, on the diameter of the disk. (Do this as closely as you can. It is not mathematically significant that this segment lies on the diameter, but let us do this for our first picture.)
2. Use Constructions → Draw Perpendicular. Click on $\overline{AB}$ then point A to draw a perpendicular line at point A.
3. Use Constructions → Draw Perpendicular. Click on $\overline{AB}$ then point B to draw a perpendicular line at point B.
4. Use Constructions → Plot Point on Object to plot a point, E on the perpendicular line through point A.
5. Use Measurements → Measure Distance to measure the distance between point E and point A.
6. Use Constructions → Draw Segment of Specific Length to plot a point, F on the perpendicular through B of the same length as the one you just measured.
7. Use Constructions → Draw Line Segment to connect the two points on the perpendiculars.

Exercise 1. *Draw a picture of your Saccheri quadrilateral. Include measurements of all sides and angles.*

Exercise 2. *Construct another Saccheri quadrilateral using the steps above, but now make a line in the first step* not *to coincide with a diameter. Draw a picture of this Saccheri quadrilateral. Include measurements of all sides and angles.*

Exploration:

Exercise 3. *Based on your observations guess the correct answers for the following questions about properties of Saccheri quadrilaterals in hyperbolic geometry:*

1. *Are summit angles of a Saccheri quadrilateral congruent?*
2. *Is each of the summit angles acute, obtuse, or right?*
3. *Is the defect of a Saccheri quadrilateral positive in the hyperbolic geometry?*

Exercise 4. *Compare the lengths of the summit and base of your Saccheri quadrilateral. What do you observe?*

Exercise 5. *Construct the diagonals of the Saccheri quadrilateral. Use Measurements → Measure Distance. Record both measurements. What do you notice? Is this similar or different from the properties of the diagonals of a rectangle in Euclidean geometry?*

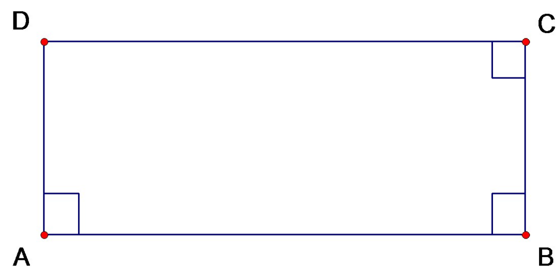
Exercise 6. *Use Constructions → Create Midpoint to find the midpoint of both the base and the summit of Saccheri Quadrilateral. The line joining the midpoints of the base and the summit is called the altitude of the Saccheri Quadrilateral. Construct the altitude of your Saccheri quadrilateral. Measure the angle between the altitude and the base and the angle between the altitude and the summit. Is the altitude, indeed, perpendicular to both the base and the summit? Is this altitude parallel to the other two other sides? Is this similar or different from the properties of an altitude in a rectangle in Euclidean Geometry?*

Exercise 7. *Write down the definition of a parallelogram. If you are not sure, check the textbook.*

Exercise 8. *Is Saccheri quadrilateral a parallelogram? Explain why or why not (Hint: recall the Alternate Interior Angles Theorem).*

3 Lambert quadrilaterals

Definition: A *Lambert quadrilateral* is a quadrilateral with three right angles. A Lambert quadrilateral looks like this:



Construction: We are going to construct a Lambert quadrilateral in the Poincaré disk applet.

1. Use Constructions → Draw Line Segment to construct segment $\overline{AB}$ along the diameter of the Poincaré disk.

2. Use Constructions → Draw Perpendicular by clicking on segment $\overline{AB}$ and then point A to construct a perpendicular through point A.

3. Use Constructions → Draw Perpendicular by clicking on segment $\overline{AB}$ and then point B to construct a perpendicular through point B.

4. Use Constructions → Plot Point on Object to construct a point E on the perpendicular line through point B.

5. Use Constructions → Draw Perpendicular and highlight the line that contains B and E and highlight the point E. This will create a perpendicular line through point E.

6. Use Constructions → Plot Intersection Point. Highlight $\overline{AC}$ and $\overline{EF}$. This creates point G. If you don't see point G, then use Edit → Move Point and move point E toward B until the intersection point appears.

Exercise 9. *Draw a picture of your Lambert quadrilateral. Include measurements of all sides and angles.*

Exercise 10. *Construct another Lambert quadrilateral using the steps above, but now make a line in the first step* not *to coincide with a diameter. Draw a picture of this Lambert quadrilateral. Include measurements of all sides and angles.*

Exploration:

Exercise 11. *Based on your observations guess the correct answers for the following questions about properties of Lambert quadrilaterals in hyperbolic geometry:*

1. *By construction there are three right angles in a Lambert quadrilateral. The fourth angle in a Lambert quadrilateral is acute, obtuse, or right?*
2. *Is the defect of a Saccheri quadrilateral positive in the hyperbolic geometry?*

Exercise 12. *Is a Lambert quadrilateral parallelogram? Explain why or why not.*

Exercise 13. *Compare the lengths of the opposite sides in a Lambert quadrilateral. Are the opposite sides equal? In a pair of opposite sides can you characterize the one which is shorter? (Hint: looks at the angles that are adjacent to the shorter side and the angles that are adjacent to the longer side)*

Exercise 14. *Draw below a picture of Saccheri quadrilateral* $\square ABFE$ *with the base* $\overline{AB}$*. Draw its altitude* $\overline{GH}$*. Are there any Lambert quadrilaterals in the picture? If yes, name them.*

Exercise 15. <u>*True or False:*</u>
(a) In every model of Euclidean Geometry every Saccheri quadrilateral is a rectangle

TRUE *FALSE*

(b) In every model of Hyperbolic Geometry every Saccheri quadrilateral is a rectangle

TRUE *FALSE*

(c) In every model of Neutral Geometry every Saccheri quadrilateral is a rectangle.

TRUE *FALSE*

(d) The answers for (a)–(c) are valid if we replace "Saccheri" with "Lambert"

TRUE *FALSE*

(e) There are no rectangles in Hyperbolic geometry.

TRUE *FALSE*

(f) There are no Saccheri quadrilaterals in Euclidean Geometry

TRUE *FALSE*

4 Homework

Now it is time to prove your observations. (Reading Section 6.7 will help)

Exercise 16. *(problem 6.26 from the book) Prove every parallelogram is a convex quadrilateral.*

Exercise 17. *(problem 6.46 from the book) Prove that if* $\square ABCD$ *is a Saccheri quadrilateral with base* $\overline{AB}$*, then*

1. *The diagonals* $\overline{AC}$ *and* $\overline{BD}$ *are congruent.*
2. *The summit angles* $\angle BCD$ *and* $\angle ADC$ *are congruent.*
3. *The segment joining the midpoint of* $\overline{AB}$ *to the midpoint of* $\overline{CD}$ *is perpendicular to both* $\overline{AB}$ *and* $\overline{CD}$*.*
4. $\square ABCD$ *is a parallelogram.*
5. $\square ABCD$ *is a convex quadrilateral.*
6. *The summit angles* $\angle BCD$ *and* $\angle ADC$ *are either right or acute.*

Exercise 18. *(problem 6.49 from the book) Prove that if* $\square ABCD$ *is a Lambert quadrilateral with right angles at vertices* A*,* B*, and* C*, then*

1. $\square ABCD$ *is a parallelogram.*
2. $\square ABCD$ *is a convex quadrilateral.*
3. $\angle ADC$ *is either right or acute.*
4. $BC \leq AD$*.*