1-D CONSERVATIVE SYSTEMS: A GEOMETRIC APPROACH

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ABSTRACT. We consider general 1-d systems of hyperbolic conservation laws with the aim of better understanding of the structure ("geometry") of such systems. For a given frame $\mathfrak{R}$ of vector fields, we derive two new overdetermined PDE systems: the $F(\mathfrak{R})$- and $\eta(\mathfrak{R})$-systems. The former gives all conservative systems with eigenframe $\mathfrak{R}$, while the latter provides extensions for these conservative systems. We also describe a recent result for non-rich, strictly hyperbolic, 3×3 conservative systems, namely: two such systems with the same eigencurves must necessarily also share Hugoniot loci.

1. **Introduction.** We consider 1-d, conservative $n \times n$-systems

$$u_t + f(u)_x = 0, \qquad t \geq 0, \, x \in \mathbb{R}, \tag{1}$$

where the unknown $u = u(t, x) = (u^1, \ldots, u^n)^T \in \mathbb{R}^n$ is the vector of conserved quantities. The smooth *flux* $f : \Omega^{\text{open}} \subset \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be hyperbolic: its Jacobian $Df(u)$ is $\mathbb{R}$-diagonalizable at each state $u \in \Omega$. In this case the eigenvectors $r_i|_u$ of $Df(u)$ form a basis for each $u \in \Omega$, and we refer to $\mathfrak{R} := \{r_1, \ldots, r_n\}$ as an *eigenframe* of (1). The system (1) is *strictly hyperbolic* provided the corresponding eigenvalues $\lambda^1(u), \ldots, \lambda^n(u)$ are distinct at each $u \in \Omega$.

While there is a well-developed theory for near-equilibrium solutions to strictly hyperbolic systems (1) (see [2]), far less is known about large variation solutions. In fact, there are systems admitting blowup solutions. While blowup of *gradients* is well-known, the blowup in [1, 3, 6] is of the solution itself in L^∞ and/or BV. Although similar phenomena had been observed earlier, the examples in [1, 3, 6] were new in that the solutions remain *uniformly* strictly hyperbolic as they explode. Also, there is nothing pathological about the flux $f(u)$ in these systems: it may be polynomial. This type of behavior is still poorly understood. In particular, it is not known if physical systems (e.g. compressible Euler) can exhibit similar behavior. The examples in [1, 3, 6] are quite special: starting from a 1-parameter family of 2×2 systems for (u^1, u^3), say, and then adding a third, decoupled conservation law for the parameter u^2, one obtains a 3×3-system (1). The same type of systems also admit space-time periodic solutions [8]. However, it now appears that the *analytic* structure of these systems is too special to be of further interest. Still, these

systems do exhibit an interesting *geometric* structure in u-space. This motivates an investigation of more general geometric issues for (1). E.g., can we prescribe curves in u-space and find systems (1) with these as *rarefaction curves* (integral curves of eigenfields)? Do the resulting systems possess extensions? Can we prescribe other types of geometric quantities for a system of conservation laws: shock curves, characteristic values, interaction coefficients, or combinations of such?

Before describing some recent results, we review the setup and findings in [4, 5]. In [4], we addressed the problem of determining the class of conservative systems (1) such that $Df(u)$ has a prescribed eigenframe $\mathfrak{R} = \{r_1, \ldots, r_n\}$. This was formulated as an algebraic-differential system, the "$\lambda(\mathfrak{R})$-system," for the corresponding eigenvalues $\lambda^1, \ldots, \lambda^n$. This λ-system was then analyzed by appealing to various integrability theorems (Frobenius, Darboux, Cartan-Kähler), and a complete breakdown of the case $n = 3$ was given. In general, once the λ's are found, the flux f can be determined by successive integration of n first order linear ODEs. It is a non-obvious fact that there are frames $\mathfrak{R}$ (see Example 2) that only admit *trivial* fluxes $f(u) = \bar{\lambda}u + c$ ($\bar{\lambda} \in \mathbb{R}$, $c \in \mathbb{R}^n$). To discuss the issue of extensions of (1) we recall the following definition and proposition [2]:

Definition 1.1. A smooth function $\eta : \Omega \to \mathbb{R}$ is an *extension* for (1) provided the map $u \mapsto \nabla\eta(u)Df(u)$ is the u-gradient of a scalar function $q : \Omega \to \mathbb{R}$. An extension η is an *entropy* for (1) provided it is a convex function of the conserved quantities, i.e. the Hessian $D^2\eta(u)$ is positive semi-definite on Ω.

Proposition 1. *Let* $\mathfrak{R} = \{r_i\}_{i=1}^n$ *be the eigenframe of (1). Then* $\eta : \Omega \to \mathbb{R}$ *is an extension for (1) if and only if*

$$\text{for each pair } 1 \leq i \neq j \leq n, \text{ either } \quad \lambda^j = \lambda^i \quad \text{ or } \quad (D^2\eta)(r_i, r_j) = 0. \qquad (2)$$

In [5], we analyzed the following question: Given a frame $\mathfrak{R}$, how large is the class of functions η which satisfy the orthogonality condition:

$$(D^2\eta)(r_i, r_j) = 0. \qquad (3)$$

Such functions provide *all* extensions for *strictly* (and a subset of all extensions for non-strictly) hyperbolic systems with eigenframe $\mathfrak{R}$. In [5], we reformulated the orthogonality condition (3) as an over-determined algebraic-differential system, the "$\beta(\mathfrak{R})$-system." The unknowns in this β-system are the lengths of the frame-vectors r_i as measured with respect to the inner-product $D^2\eta$. The Hessian $D^2\eta$ is determined by these lengths and the given frame $\mathfrak{R}$. In turn, the extensions η can be determined, in principle, by integration of $n(n + 1)$ first order linear ODEs.

The results in [4, 5] provide information about *how many* conservative systems there are with a given eigenframe, and *how many* extensions these systems are equipped with. This information is given in terms of the number of free parameters or functions that determine a general solution of the λ- and β-systems. E.g., for the 3×3-case there are only two possibilities if the frame admits strictly hyperbolic systems: either the resulting systems are all rich[1] or they form a 1-parameter family (up to addition of a trivial flux). The latter class is more interesting one: it covers systems with eigenframe that of the full Euler system, as well as the blowup examples described above.

While the analysis in [4, 5] gives the sizes of the solutions sets for the λ- and β-systems, it does not explicitly provide the fluxes and their extensions. As noted above this requires additional, and potentially challenging, ODE integrations.

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[1]We refer to [2] for this and other standard terminology.

In this paper, we address this practical drawback by deriving two overdetermined, but purely differential, systems $f(\mathfrak{R})$ and $\eta(\mathfrak{R})$, whose solutions provide *directly* the fluxes $f(u)$ with eigenframe $\mathfrak{R}$, and their associated extensions $\eta(u)$. For the former we use a coordinate-free notion of *Jacobian maps*[2] and for the latter we use a coordinate-free notion of *Hessian metrics*[3]. By solving the $f(\mathfrak{R})$- and $\eta(\mathfrak{R})$-systems we avoid the additional integrations required to reconstruct fluxes and extensions from the solutions of the λ- and β-systems. Not surprisingly, the degrees of freedom in the general solutions to the $f(\mathfrak{R})$- and $\eta(\mathfrak{R})$-systems are determined by the degrees of freedom in the general solution the $\lambda(\mathfrak{R})$- and $\beta(\mathfrak{R})$-systems, respectively. In fact, the latter systems provide exactly the compatibility conditions for the former overdetermined systems.

After reviewing some background on frames and connections in Section 2, we derive the $f(\mathfrak{R})$- and $\eta(\mathfrak{R})$-systems and establish their relationships with the $\lambda(\mathfrak{R})$- and $\beta(\mathfrak{R})$-systems in Sections 3 and 4. Section 5 provides some examples. Finally, in Section 6, we describe a recent result relating Hugoniot loci and eigenframes.

2. Frames and connections.

2.1. Frames and coframes.
Let $\Omega \subset \mathbb{R}^n$ be an open set. Denote by $\mathcal{X}(\Omega)$ and $\mathcal{X}^*(\Omega)$ the sets of smooth vector fields and smooth one-forms on $\Omega \subset \mathbb{R}^n$, respectively. A *frame* $\{r_1, \ldots, r_n\}$ is a set of smooth vector fields which are linearly independent at each point $u \in \Omega$. A *coframe* $\{\ell^1, \ldots, \ell^n\}$ is a set of smooth one-forms which are linearly independent at each point $u \in \Omega$. If $\ell^i(r_j) = \delta^i_j$ (Kronecker delta), then the coframe and frame are *dual*. Given coordinates $(u^1, \ldots, u^n)$ on Ω, the associated coordinate frame is $\left\{\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\right\}$, with dual coframe $\{du^1, \ldots, du^n\}$. For any frame $\mathfrak{R} = \{r_i\}_{i=1}^n$, its *structure coefficients* c^k_{ij} are defined by $[r_i, r_j] = \sum_k c^k_{ij} r_k$, where $[\cdot, \cdot]$ denotes the commutator.

2.2. Connections.
An *affine connection* ∇ on Ω is an $\mathbb{R}$-bilinear map

$$\mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathcal{X}(\Omega) \qquad (X, Y) \mapsto \nabla_X Y$$

such that for any smooth function h on Ω

$$\nabla_{hX} Y = h \nabla_X Y, \qquad \nabla_X(h\,Y) = (Xh)Y + h\,\nabla_X Y. \tag{4}$$

A connection is uniquely defined by prescribing it on a frame:

$$\nabla_{r_i} r_j = \sum_{k=1}^n \Gamma^k_{ij} r_k, \tag{5}$$

where the coefficients Γ^k_{ij} are the *connection components* relative to $\mathfrak{R}$. The connection has a unique extension to an $\mathbb{R}$-bilinear map $\mathcal{X}(\Omega) \times \mathcal{T}(\Omega) \to \mathcal{T}(\Omega)$, where $\mathcal{T}(\Omega)$ is a set of all tensor products of $\mathcal{X}(\Omega)$ and $\mathcal{X}^*(\Omega)$ (see [7]). We record the following properties of this extension:

$$\nabla_X h = X(h), \ \ \nabla_X(h\omega) = (Xh)\omega + f\nabla_X \omega, \ \ X(\omega(Y)) = [\nabla_X \omega](Y) + \omega(\nabla_X Y), \tag{6}$$

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[2]The notion of a Jacobian map was introduced in Remark 2.14 of [5] to provide an alternative, coordinate-free derivation of the λ-system. It is different from the *differential* of a map, which in coordinates, is also given by a Jacobian matrix.

[3] The notion of a Hessian metric has been extensively studied [7]. In Remark 2.15 of [5], it was used to provide an alternative, coordinate-free derivation of the β-system.

where h is a function, ω is a one-form, and X, Y are vector-fields on Ω. Note that (5)-(6) imply

$$\nabla_{r_i} \ell_j = - \sum_{k=1}^{n} \Gamma_{ij}^{k} \, \ell_k. \tag{7}$$

We now fix the connection ∇ on Ω by requiring the state variables $\{u_i\}_{i=1}^{n}$ of (1) to be *affine coordinates* on Ω, i.e. we impose

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0 \qquad \text{for all } i, j = 1, \ldots, n. \tag{8}$$

This connection has the following two important properties:

$$c_{jk}^{i} = \Gamma_{jk}^{i} - \Gamma_{kj}^{i} \qquad \text{(Symmetry)}, \tag{9}$$

where c's and Γ's are structure and connection coefficients of a frame, and

$$\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X = \nabla_{[X,Y]} \qquad \text{(Flatness)}, \tag{10}$$

where $X, Y \in \mathcal{X}(\Omega)$. Finally, we observe that if

$$r_i = \sum_{k=1}^{n} R_i^k \frac{\partial}{\partial u^k} \qquad \text{and} \qquad \ell^i = \sum_{k=1}^{n} L_i^k du^k \qquad (i = 1, \ldots, n) \tag{11}$$

then (4) and (8) imply $\Gamma_{ij}^{k}(u) = L^k(D_u R_j) R_i$.

3. **The $f(\mathfrak{R})$-system.** We fix a frame $\mathfrak{R} = \{r_i\}_{i=1}^{n}$ on Ω and ask for the class of hyperbolic systems (1) whose flux-Jacobians Df has eigenframe $\mathfrak{R}$. This was formulated in [4] as an overdetermined algebraic-differential system for the eigenvalues $\lambda^i(u)$. To derive a purely differential system, which *directly* yields the components of the flux f, we apply:

Proposition 2. *Let $\{u_i\}_{i=1}^{n}$ be affine coordinates on a simply connected open subset $\Omega \subset \mathbb{R}^n$, and $f = (f^1, \ldots, f^n) : \Omega \to \mathbb{R}^n$ be a smooth map. Then $r_i|_u$ is an eigenvector of the Jacobian matrix $Df(u)$ if and only if there is a function $\lambda^i \colon \Omega \to \mathbb{R}$ with*

$$\nabla_{r_i} F = \lambda^i \, r_i, \tag{12}$$

where the vector field F is defined by $F := f^1 \frac{\partial}{\partial u^1} + \cdots + f^n \frac{\partial}{\partial u^n}$. If (12) is satisfied, then $\lambda^i(u)$ is the corresponding eigenvalue of $Df(u)$.

Hence, we seek a vector field $F \in \mathcal{X}(\Omega)$ such that (12) is satisfied for $i = 1, \ldots, n$. We write $F = \sum_{i=1}^{n} \widetilde{f}^i r_i$. Then, using (4), we find that (12) is equivalent to

$$\sum_{j=1}^{n} \left(r_i(\widetilde{f}^j) + \sum_{k=1}^{n} \widetilde{f}^k \Gamma_{ik}^{j} \right) r_j = \lambda^i \, r_i. \tag{13}$$

Collecting the coefficients, we obtain the systems

$$r_i(\widetilde{f}^j) + \sum_{k=1}^{n} \widetilde{f}^k \, \Gamma_{ik}^{j} = 0 \qquad \text{for } i \neq j, \qquad \text{and} \qquad \lambda^i = r_i(\widetilde{f}^i) + \sum_{k=1}^{n} \widetilde{f}^k \Gamma_{ik}^{i} \,. \tag{14}$$

The $f(\mathfrak{R})$-*system* $(14)_1$ consists of $n^2 - n$ linear 1st order PDEs in n unknowns $\{\widetilde{f}^i\}_{i=1}^{n}$. For a given solution of $(14)_1$, the flux f is determined by $f = R\,(\widetilde{f}^1, \ldots, \widetilde{f}^n)^T$, where the entries of matrix R are given by $(11)_1$. Finally, the eignevalues λ^i of Df are given by $(14)_2$.

As $(14)_1$ is overdetermined for $n > 2$, the existence of solutions is not guaranteed. We now derive necessary and sufficient conditions for the existence of solutions. First, flatness of the connection (10) implies that

$$\nabla_{r_i} \circ \nabla_{r_j} F - \nabla_{r_j} \circ \nabla_{r_i} F = \nabla_{[r_i, r_j]} F = \sum_k c_{ij}^k \nabla_{r_k} F, \quad (i \neq j). \qquad (15)$$

Substitution of (12) into (15) gives

$$\nabla_{r_i}\left(\lambda^j r_j\right) - \nabla_{r_j}\left(\lambda^i r_i\right) = \sum_k c_{ij}^k \lambda^k r_k. \qquad (16)$$

By expanding (16) using (4), and then collecting the coefficients of the r_i, we obtain the following algebraic-differential system for the λ^i:

$$r_i(\lambda^j) = \Gamma_{ji}^j (\lambda^i - \lambda^j), \qquad \qquad \text{for } i \neq j, \qquad (17)$$

$$(\lambda^i - \lambda^k)\Gamma_{ji}^k = (\lambda^j - \lambda^k)\Gamma_{ij}^k, \qquad \text{for all distinct } i, j \text{ and } k. \qquad (18)$$

This is exactly the $\lambda(\mathfrak{R})$-system derived in $[4]$. Therefore, a necessary condition for existence of a solution to the $f(\mathfrak{R})$-system is the existence of a solution to the $\lambda(\mathfrak{R})$-system. Vice versa, by substituting a solution $\{\lambda^i\}_{i=1}^n$ of the $\lambda(\mathfrak{R})$-system into $(14)_2$, the combined system $(14)_{1,2}$ is a system of Frobenius type.[4] A direct computation shows that this system is compatible. By Frobenius' Theorem it follows that for each solution $\{\lambda^i\}_{i=1}^n$ of the λ-system, and a choice of values $\widetilde{f}^1(\bar{u}), \dots, \widetilde{f}^n(\bar{u})$ at some $\bar{u} \in \Omega$, there is a unique solution of $(14)_1$, that satisfies $(14)_2$.

4. **The $\eta(\mathfrak{R})$-system.** For a fixed frame $\mathfrak{R} = \{r_i\}_{i=1}^n$ on Ω, we now ask for the class of extensions for systems (1) whose flux-Jacobians Df have eigenframe $\mathfrak{R}$. By restricting to extensions η satisfying (3), this was formulated in $[5]$ as an overdetermined algebraic-differential system for the lengths $\beta^i(u)$. To derive a purely differential system, which *directly* yields the extensions of (1), we apply:

Proposition 3. *Let $\{u_i\}_{i=1}^n$ be affine coordinates on a simply connected open subset $\Omega \subset \mathbb{R}^n$, and $\eta \colon \Omega \to \mathbb{R}$. Then $\{r_i|_u\}_{i=1}^n$ are orthogonal with respect to the inner product defined by the Hessian $D_u^2 \eta(u)$ if and only if there exist functions $\beta^i \colon \Omega \to \mathbb{R}$, $i = 1, \dots, n$, such that*

$$\nabla_{r_i} d\eta = \beta^i \ell^i, \quad \text{for all } i = 1, \dots, n, \qquad (19)$$

where $\{\ell^1, \dots, \ell^n\}$ is the dual coframe to $\mathfrak{R}$. If (19) is satisfied, then β^i is the length of the vector $r_i|_u$ relative to the Hessian inner product $D_u^2 \eta$.

Hence, we seek $\eta : \Omega \to \mathbb{R}^n$ such that (19) is satisfied for some functions β^i. Since $d\eta = \sum_j r_j(\eta)\ell^j$, and using (6)-(7), we obtain

$$\nabla_{r_i} d\eta = \sum_j \left(r_i(r_j(\eta)) - \sum_k r_k(\eta)\Gamma_{ij}^k\right)\ell^j. \qquad (20)$$

Substituting (20) into (19) and collecting coefficients of ℓ^i, we obtain the systems

$$r_i(r_j(\eta)) - \sum_k \Gamma_{ij}^k r_k(\eta) = 0 \quad \text{for } i \neq j, \quad \text{and} \quad \beta^i = r_i(r_i(\eta)) - \sum_k \Gamma_{ii}^k r_k(\eta). \quad (21)$$

The $\eta(\mathfrak{R})$-*system* $(21)_1$ consists of $n(n-1)$ 2nd order PDEs for the single unknown η. For a given solution of $(21)_1$, the lengths β^i of the vectors $r_i|_u$ relative to the inner product $D_u^2 \eta$ are given by $(21)_2$.

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[4]See $[4]$ for Frobenius type systems and Frobenius' Theorem adapted to the present context.

We next derive necessary and sufficient conditions for the existence of solutions to the overdetermined $\eta(\mathfrak{R})$-system. Again, the flatness condition (10) implies

$$\nabla_{r_i} \nabla_{r_j} d\eta - \nabla_{r_j} \nabla_{r_j} d\eta = \nabla_{[r_i, r_j]} d\eta = \sum_k c_{ij}^k \nabla_{r_k} d\eta. \quad (i \neq j) \qquad (22)$$

Substitution of (19) into (22) gives

$$\nabla_{r_i}(\beta_j \ell^j) - \nabla_{r_j}(\beta_i \ell^i) = \sum_k c_{ij}^k \beta_k \ell^k. \qquad (23)$$

By expanding (23) using (7), and then collecting the coefficients of ℓ^i, we obtain the following algebraic-differential system for the β^i:

$$r_i(\beta_j) = \beta_j(\Gamma_{ij}^j + c_{ij}^j) - \beta^i \Gamma_{jj}^i, \qquad\qquad \text{for } i \neq j, \qquad (24)$$

$$\beta_k c_{ij}^k + \beta_j \Gamma_{ik}^j - \beta_i \Gamma_{jk}^i = 0 \qquad\qquad \text{for } i < j, i, j \neq k. \qquad (25)$$

This is exactly the $\beta(\mathfrak{R})$-system derived in [5]. Thus, a necessary condition for the existence of a solution to the $\eta(\mathfrak{R})$-system is the existence of a solution to the $\beta(\mathfrak{R})$-system. Vice versa, given a solution $\{\beta^i\}_{i=1}^n$ of the $\beta(\mathfrak{R})$-system, we introduce $\widetilde{\eta}_i := r_i(\eta)$ and rewrite $(21)_{1,2}$ as a first order Frobenius type system in the n unknowns $\tilde{\eta}_i$:

$$r_i(\widetilde{\eta}_j) - \sum_k \Gamma_{ij}^k \widetilde{\eta}_k = 0 \quad \text{for } i \neq j, \quad \text{and} \quad r_i(\widetilde{\eta}_i) - \sum_k \Gamma_{ii}^k \widetilde{\eta}_k = \beta^i. \qquad (26)$$

A direct computation shows that this is a compatible system. By Frobenius' Theorem, we conclude that any solution $\{\beta^i\}_{i=1}^n$ of the $\beta(\mathfrak{R})$-system, together with a choice of values $\widetilde{\eta}_1(\bar{u}), \ldots, \widetilde{\eta}_n(\bar{u})$ at some point $\bar{u} \in \Omega$, provide a unique solution of the system $(26)_1$ satisfying $(26)_2$. Finally, if $\{\widetilde{\eta}_i\}_{i=1}^n$ solve $(26)_1$-$(26)_2$, then $(\eta_1, \ldots, \eta_n) := (\widetilde{\eta}_1, \ldots, \widetilde{\eta}_n) R^{-1}$ (R given by (11)), provide a solution η of the combined system $(21)_{1,2}$ via $\eta_i = \frac{\partial \eta}{\partial u_i}$.

5. **Examples.** We include some representative examples where we use Maple code developed by the authors[5] to calculate explicit fluxes and their extensions from a given eigenframe. Cf. Examples 6.5, 6.6, 6.8 and 6.13 in [5].

Example 1. *Consider a rich orthogonal frame:* $r_1 = (u^1, u^2, 0)^T$, $r_2 = (-u^2, u^1, 0)^T$, $r_3 = (0, 0, 1)^T$. *Solving the* $f(\mathfrak{R})$-*system, we obtain the family of fluxes depending on three arbitrary functions of one-variable:*

$$f(u) = \begin{pmatrix} -u^1 \int_*^{u^1} \dfrac{F_2\left(\frac{1}{a}\sqrt{(u^1)^2 + (u^2)^2 - a^2}\right)}{a\sqrt{(u^1)^2 + (u^2)^2 - a^2}}\, da + u^1 F_1\left((u^1)^2 + (u^2)^2\right) - F_1\left(\frac{u^2}{u^1}\right)\frac{u^2}{u^2} \\ -u^2 \int_*^{u^1} \dfrac{F_2\left(\frac{1}{a}\sqrt{(u^1)^2 + (u^2)^2 - a^2}\right)}{a\sqrt{(u^1)^2 + (u^2)^2 - a^2}}\, da + u^2 F_1\left((u^1)^2 + (u^2)^2\right) - F_1\left(\frac{u^2}{u^1}\right) \\ F_3(u^3) \end{pmatrix}.$$

Solving the $\eta(\mathfrak{R})$-*system, we obtain the family of extensions also depending on three arbitrary functions of one-variable:*

$$\eta(u) = \int_*^{u^1} \dfrac{G_2\left(\frac{1}{a}\sqrt{(u^1)^2 + (u^2)^2 - a^2}\right)}{a\sqrt{(u^1)^2 + (u^2)^2 - a^2}}\, da + G_1\left((u_1)^2 + (u^2)^2\right) + G_3(u^2). \qquad (27)$$

There are strictly hyperbolic systems in this class, and any extension of such a system is given by (27). For this frame, it is much harder (and beyond the capability

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of MAPLE*) to obtain fluxes and extensions if we first solve the $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$-systems, and then try to find f and η by solving ODE's. It is not difficult, however, to find λ's and β's, after we found f and η, using, for instance, $(14)_2$ and $(21)_2$.

Example 2. *For the rich frame* $r_1 = (u^1, u^2, 0)^T$, $r_2 = (-u^2, u^1, 0)^T$, $r_3 = (-u^2, u^1, 1)^T$, *we solve the* $f(\mathfrak{R})$-*system and obtain only trivial fluxes:*

$$f(u) = c_1(u^1, u^2, u^3)^T + (c_2, c_3, c_4)^T,$$

with $\lambda^1 = \lambda^2 = \lambda^3 = c_1 \in \mathbb{R}$. *The* $\eta(\mathfrak{R})$-*system provides a family of extensions that satisfy the orthogonality condition* (3): $\eta(u) = a_1 u^1 + a_2 u^2 + G(u^3)$. *Obviously, any function* $\eta(u)$ *satisfies the full condition* (2) *and, hence, is an extension.*

Example 3. *For the non-rich frame* $r_1 = (-1, 0, u^2 + 1)^T$, $r_2 = \left(\frac{u^3}{(u^2)^2 - 1}, -1, u^1\right)^T$, $r_3 = (1, 0, 1 - u^2)^T$, *we solve the* $f(\mathfrak{R})$-*system and obtain a family of fluxes:*

$$f(u) = c_1 \begin{pmatrix} u^3 - u^1 u^2 - u^1 \\ \frac{1}{2}(u^2)^2 - u^2 \\ (1 + u^2)(u^1 - u^1 u^2 - u^3) \end{pmatrix} + c_2 \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} + \begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix},$$

which is, up to a trivial flux, depends on one constant. Using $(14)_2$, *we find that* $\lambda^1 = c_2 - 2 c_1$, $\lambda^2 = c_2 + c_1(u^2 - 1)$ *and* $\lambda^3 = c_2$. *Observe that there* are *strictly hyperbolic system with this frame. Solving the* $\eta(\mathfrak{R})$-*system, we obtain a family of extensions*

$$\eta(u) = a_1 u^1 + a_3 u^3 + G(u^2),$$

depending on one arbitrary function. These are all the possible extensions of a strictly hyperbolic system with the given frame. From $(21)_2$, *we find that* $\beta^1 = \beta^3 \equiv 0$ *and* $\beta_2 = G''(u^2)$.

Example 4. *For the non-rich frame* $r_1 = (1, u^2, 0)^T$, $r_2 = (u^3, 1, 0)^T$, $r_3 = (0, 0, 1)^T$, *we solve the* $f(\mathfrak{R})$-*system and obtain a family of fluxes:*

$$f(u) = c_1 \left(u^1, u^2, F(u^3)\right)^T + (c_2, c_3, c_4)^T,$$

which is, up to a trivial flux, depends on one arbitrary function of one variable. Using $(14)_2$, *we find that* $\lambda^1 = \lambda^2 = c_1$ *and* $\lambda^3 = F'(u_3)$. *Therefore, there are no strictly hyperbolic system with this frame. Solving the* $\eta(\mathfrak{R})$-*system, we obtain a family of extensions:*

$$\eta(u) = a_1 u^1 + a_2 u^2 + G(u^3).$$

which satisfy the orthogonality condition (3). *Note, however, that there may be other extensions that satisfy condition* (2) *due to equality of eigenvalues* $\lambda^1 = \lambda^2$.

6. **On Hugoniot loci and eigencurves for** 3×3**-systems.** Together with [4, 5], the analysis above provides detailed information about systems (1) with a given eigenframe: the $f(\mathfrak{R})$- and $\eta(\mathfrak{R})$-systems provide the fluxes and entropies, while the $\lambda(\mathfrak{R})$- and $\beta(\mathfrak{R})$-systems tell us how many such there are. Note that prescribing $\mathfrak{R}$ amounts to prescribing *rarefaction curves*, and these make up one part of the wave curves. The other part is the *Hugoniot locus* $\mathcal{H}$. For a given flux f and state $\bar{u} \in \Omega$,

$$\mathcal{H}_{f,\bar{u}} := \{u \in \Omega \,|\, \exists\, s \in \mathfrak{R} \text{ such that } s(u - \bar{u}) = f(u) - f(\bar{u})\}.$$

To complement our analysis above, it would seem natural to ask: "Given a family of curves in Ω, what is the set if systems (1) with these as Hugoniot loci?" However, this is not a well-formulated problem: due to the symmetry relation "$\bar{v} \in \mathcal{H}_{f,\bar{u}} \Leftrightarrow \bar{u} \in \mathcal{H}_{f,\bar{v}}$" it is hard to prescribe a family of curves that *could* be Hugoniot loci. Unless, of course, one starts with the Hugoniot loci of a given system (1). Thus, the

natural complement to our earlier analysis is to ask: "Given a hyperbolic system (1), what other systems have the same Hugoniot loci?"

We shall describe a partial result for 3×3-systems in this direction. First, there are simple examples showing that conservation laws with the same eigenframe does not, in general, share the same Hugoniot loci. This is not surprising since $\mathcal{H}_{f,\bar{u}}$ depends on all of f, and not only its eigenframe. However, we have the following:

Proposition 4. *Consider a strictly hyperbolic, non-rich 3×3-system (1). Then all systems with the same eigenframe also have the same Hugoniot loci.*

Proof. Let $\mathfrak{R}$ be the eigenframe and g the flux corresponding to our system. The case of a strictly hyperbolic non-rich system only occurs in the IIa subcase from [4]. The proof of Theorem 3.2 in that paper invokes the Frobenius Integrability Theorem to show that for any choice of the initial conditions $\lambda^2(\bar{u})$ and $\lambda^3(\bar{u})$ for two of the eigenvalues, there is a unique Jacobian with the eigenframe $\mathfrak{R}$. Let A be the Jacobian with eigenvalues satisfying initial condition $\lambda^2(\bar{u}) = 1$ and $\lambda^3(\bar{u}) = 0$. Let a_2 and a_3 be the eigenvalues of $Dg(\bar{u})$ corresponding to r_2 and r_3. A direct computation show that $B = (a_2 - a_3) A + a_3 I$ is a Jacobian with with eigenframe $\mathfrak{R}$ and eigenvalues satisfying initial conditions $\lambda^2(\bar{u}) = a_2$ and $\lambda^3(\bar{u}) = a_2$. By the proof of Theorem 3.2 in [4] it is the unique such Jacobian.

Thus the flux g must be of the form $g(u) := (a_2 - a_3) f(u) + a_3 u + \bar{v}$, where $f(u)$ is a flux with the Jacobian A, and $\bar{v} \in \mathbb{R}^n$ is a fixed vector. Note that if $f(u) - f(v) = s(u - v)$, then $g(u) - g(v) = [s(a_2 - a_3) + a_3](u - v)$ and thus the Hugoniot loci for f are Hugoniot loci for g. Since $a_2 \neq a_3$, we can also write $f(u) = b_1 g(u) + b_2 u + \bar{v}'$ for some b_1, b_2, and $\bar{v}'$, and determine by the same argument that also the Hugoniot loci for g are Hugoniot loci for f. Thus all systems with this eigenframe have the same Hugoniot loci. $\qquad\square$

To appreciate the relevance of this result, we recall (see Introduction) that given frame in $\mathbb{R}^3$, there are only two possibilities for it to admit strictly hyperbolic systems: either these are all rich, or they form a 1-parameter family (up to a trivial flux). Thus, Proposition 4 applies to all systems in the latter 1-parameter class. In particular, it applies to any system with eigenframe that of the full Euler system, or that of the blowup examples in [1, 3, 6].

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