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Construction of Conservative Systems

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Abstract. We consider the problem of constructing systems of hyperbolic conservation laws in one space dimension with prescribed eigencurves. This yields an overdetermined system of equations for the eigenvalues-to-be. These equations are analyzed with techniques from exterior differential systems.

1. Introduction

Consider a system of n conservation laws in one space dimension written in canonical form,

$$u_t + f(u)_x = 0\,. \tag{1.1}$$

Here the unknown state $u = u(t, x) \in \mathbb{R}^n$ is assumed to take values in some open subset $\Omega \subset \mathbb{R}^n$ and the flux f is a nonlinear map from Ω into $\mathbb{R}^n$. The eigenvalues and eigenvectors of the Jacobian matrix $Df(u)$ provide information that is used to solve the Cauchy problem for (1.1). In particular, the geometric properties of the integral curves of eigenvector fields of Df play an important role. Together with the Hugoniot locus they form wave curves that are used to build solutions of (1.1).

We are interested in what freedom one has in *prescribing* such eigenfields. Given n vector fields, we want to determine if there are any conservative systems (1.1) with the property that the given vector fields are the eigenfields of $Df(u)$. If there exist such systems we are also interested in knowing how many there are.

As an example consider the Euler equations for one-dimensional flow of a compressible gas with a given pressure function. The pressure determines the eigenfields, and it turns out (see Example 5.1) that there is a two-parameter family of conservative systems with the same eigencurves as the given Euler system.

We will formulate the problem as an algebraic-differential system that the eigenvalues-to-be must satisfy. This "λ-system" is a linear, homogeneous, and overdetermined system (for $n \geq 3$) that can be analyzed by methods from exterior differential systems (Cartan-Kähler theory, [**BCG^3**], [**IL**]). It turns out that, apart from two extreme cases (see Section 4.2), the structure of the set of solutions

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to the λ-system is, in general, rather involved. Already the case $n = 3$ allows a number of different possibilities (see Section 4.4 and the examples in Section 5). Before giving a precise formulation we review some relevant concepts and results.

2. Preliminaries and problem formulation

2.1. Eigenvectors and eigencurves. We consider *hyperbolic* systems of conservation laws (1.1) where $Df(u)$ is real-diagonalizable for each state $u \in \Omega$. *Strict* hyperbolicity means that the eigenvalues $\lambda^i(u)$ of $Df(u)$ are real and distinct:

$$\lambda^1(u) < \cdots < \lambda^n(u)\,, \qquad \forall\, u \in \Omega\,. \tag{2.1}$$

The corresponding right and left eigenvectors $R_i(u)$ (columns) and $L^i(u)$ (rows) of $Df(u)$ are denoted by

$$R_i(u) = \left[R_i^1(u), \ldots, R_i^n(u)\right]^T, \qquad L^i(u) = \left[L_1^i(u), \ldots, L_n^i(u)\right].$$

(A superscript T denotes transpose.) We refer to the $R_i(u)$ as the *eigenfields* and their integral curves in u-space as *eigencurves*. Diagonalizing Df we have

$$Df(u) = R(u)\,\Lambda(u)\,L(u)\,, \tag{2.2}$$

where

$$R(u) = [R_1(u)\,|\,\cdots\,|\,R_n(u)]\,, \quad \Lambda(u) = \mathrm{diag}[\lambda^1(u), \ldots, \lambda^n(u)]\,,$$

and

$$L(u) = R(u)^{-1} = \begin{bmatrix} L^1(u) \\ \hline \vdots \\ \hline L^n(u) \end{bmatrix}.$$

2.2. Connection on the frame bundle. Given an n-dimensional smooth manifold M we let $\mathcal{X}(M)$ and $\mathcal{X}^*(M)$ denote the set of vector fields and differential 1-forms on M, respectively. A *frame* $\{r_1, \ldots, r_n\}$ is a set of vector fields which span the tangent space T_pM at each point $p \in M$. A *coframe* $\{\ell^1, \ldots, \ell^n\}$ is a set of n differential 1-forms which span the cotangent space T_p^*M at each point $p \in M$. The coframe and frame are *dual* if $\ell^i(r_j) = \delta_j^i$ (Kronecker delta). If $u^1, \ldots, u^n$ are local coordinate functions on M, then $\{\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\}$ is the corresponding local *coordinate frame*, while $\{du^1, \ldots, du^n\}$ is the dual local *coordinate coframe*. For a given frame $\{r_1, \ldots, r_n\}$ the *structure coefficients* c_{ij}^k are defined through

$$[r_i, r_j] = \sum_{k=1}^n c_{ij}^k\, r_k\,, \tag{2.3}$$

and the dual coframe has related structure equations given by

$$d\ell^k = -\sum_{i<j} c_{ij}^k\, \ell^i \wedge \ell^j\,. \tag{2.4}$$

It can be shown that there exist coordinate functions $w^1, \ldots, w^n$ on Ω such that $r_i = \frac{\partial}{\partial w^i}$, $i = 1 \ldots, n$, if and only if $r_1, \ldots, r_n$ commute, i.e. all structure coefficients are zero. Next, an *affine connection* ∇ on M is an $\mathbb{R}$-bilinear map

$$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \qquad (X, Y) \mapsto \nabla_X Y$$

such that for any smooth function f on M

$$\nabla_{fX} Y = f\nabla_X Y, \qquad \nabla_X(fY) = (Xf)Y + f\nabla_X Y\,. \tag{2.5}$$

By $\mathbb{R}$-bilinearity and (2.5) the connection is uniquely defined by prescribing it on a frame:

$$\nabla_{r_i} r_j = \sum_{k=1}^{n} \Gamma^k_{ij} r_k,$$

where the smooth coefficients Γ^k_{ij} are called *connection components, or Christoffel symbols,* relative to the frame $\{r_1, \ldots, r_n\}$. Any choice of a frame and n^3 functions Γ^k_{ij}, $i, j, k = 1, \ldots, n$, defines an affine connection on M. A change of frame induces a change of the connection components, and this change is not tensorial. E.g., a connection with zero components relative to a coordinate frame, may have non-zero components relative to a non-coordinate frame.

Given a frame $\{r_1, \ldots, r_n\}$ with associated Christoffel symbols Γ^k_{ij} and dual frame $\{\ell^1, \ldots, \ell^n\}$, we define the *connection 1-forms* μ^j_i by

$$\mu^j_i := \sum_{k=1}^{n} \Gamma^j_{ki} \ell^k .$$

In turn, these are used to define two important tensor-fields: the *torsion* 2-forms

$$\mathbf{T}^i := d\ell^i + \sum_{k=1}^{n} \mu^i_k \wedge \ell^k = \sum_{k<m} T^i_{km} \ell^k \wedge \ell^m, \qquad i = 1, \ldots, n\,, \tag{2.6}$$

and the *curvature* 2-forms

$$\mathbf{R}^j_i := d\mu^j_i + \sum_{k=1}^{n} \mu^j_k \wedge \mu^k_i = \sum_{k<m} R^j_{i\,km} \ell^k \wedge \ell^m\,. \tag{2.7}$$

Here

$$T^i_{km} = \Gamma^i_{km} - \Gamma^i_{mk} - c^i_{km} \tag{2.8}$$

$$R^j_{i\,km} = r_k\big(\Gamma^j_{mi}\big) - r_m\big(\Gamma^j_{ki}\big) + \sum_s \big(\Gamma^j_{ks}\Gamma^s_{mi} - \Gamma^j_{ms}\Gamma^s_{ki} - c^s_{km}\Gamma^j_{si}\big) \tag{2.9}$$

are components of the torsion and curvature tensors respectively, and these *do* change tensorially under a change of frame. We can write equations (2.6) and (2.7) in the compact matrix form

$$\mathbf{T} = d\ell + \mu \wedge \ell, \quad \mathbf{R} = d\mu + \mu \wedge \mu \tag{2.10}$$

where $\ell = (\ell^1, \ldots, \ell^n)^T$, $\mathbf{T} = (\mathbf{T}^1, \ldots, \mathbf{T}^n)^T$, and $\mathbf{R}$ and μ are the matrices with components $\mathbf{R}^j_i$ and μ^j_i respectively. The connection is called *symmetric* if the torsion form is identically zero and it is called *flat* if the curvature form is identically zero. Equivalently:

$$d\ell = -\mu \wedge \ell \quad \text{(Symmetry)}, \qquad d\mu = -\mu \wedge \mu \quad \text{(Flatness)}. \tag{2.11}$$

In terms of Christoffel symbols and structure coefficients this is equivalent to

$$c^i_{km} = \Gamma^i_{km} - \Gamma^i_{mk} \qquad \text{(Symmetry)} \tag{2.12}$$

$$r_m\big(\Gamma^j_{ki}\big) - r_k\big(\Gamma^j_{mi}\big) = \sum_s \big(\Gamma^j_{ks}\Gamma^s_{mi} - \Gamma^j_{ms}\Gamma^s_{ki} - c^s_{km}\Gamma^j_{si}\big) \quad \text{(Flatness)}. \tag{2.13}$$

One can also show that a connection ∇ is symmetric and flat if and only if in a neighborhood of each point there exist coordinate functions $u^1, \ldots, u^n$ with the property that the Christoffel symbols relative to the coordinate frame are zero: $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$ for all $i, j = 1, \ldots, n$.

2.3. Formulation of problem. We shall consider the "inverse" problem of finding systems of conservation laws with prescribed "geometric properties". One may formulate different problems of this type. The most direct formulation is obtained by prescribing the eigencurves, and this is what we do here.

We fix a base point $\bar{u}$ in an open set $\Omega \subset \mathbb{R}^n$ which is smoothly contractible to a point. Throughout $u = (u^1, \ldots, u^n)$ will denote a *fixed* coordinate system on a neighborhood of $\bar{u}$.

Remark 2.1. Below we will formulate a system of PDEs which expresses that a certain matrix is a Jacobian matrix with respect to the u coordinates. The property of being a Jacobian with respect to a system of coordinates is not invariant under most changes of coordinates. This is the reason why we need to fix the coordinate system at the outset.

Next we assume that we are given n linearly independent column n-vectors $R_i(u)$ on Ω, and we define

$$R(u) := [R_1(u) \,|\, \cdots \,|\, R_n(u)], \qquad \text{and} \qquad L(u) := R(u)^{-1} := \begin{bmatrix} L^1(u) \\ \hline \vdots \\ \hline L^n(u) \end{bmatrix}. \tag{2.14}$$

The proofs of the results make use of the Cauchy-Kowalevskaya theorem and the more general Cartan-Kähler theorem. These theorems requires analytic data and we therefore assume that $R(u)$, and hence $L(u)$, are real analytic in Ω. We can now formulate our problem:

Problem 1. *Find n real valued functions $\lambda^1(u), \ldots, \lambda^n(u)$ defined on some neighborhood $\mathcal{U}$ of $\bar{u}$ such that, with $\Lambda(u) := \mathrm{diag}[\lambda^1(u), \ldots, \lambda^n(u)]$, the matrix*

$$A(u) := R(u)\Lambda(u)L(u) \tag{2.15}$$

is the Jacobian matrix of a map $f : \mathcal{U} \to \mathbb{R}^n$.

2.4. Related works. Problem 1 was addressed by Dafermos [**D1**] for 2×2-systems in several space dimensions under the further requirement that the Jacobians in the various spatial directions commute. In [**D1**] it was shown how to construct such systems for any pair of linearly independent vector fields. The special case of one-dimensional 2×2-systems is considered briefly below (Section 4.3).

Sévennec [**Sev**] has characterized the quasilinear systems

$$v_t + A(v)v_x = 0$$

that can be transformed to conservative form (1.1) by a change of dependent variables $u = \phi(v)$. This characterization involves a version of what we refer to as the λ-system, see (3.10)-(3.11).

The class of *rich* systems has been studied by many authors. These are systems equipped with a coordinate system of Riemann coordinates, see [**D2**], [**Ser**]. Conlon and Liu [**CL**] considered rich systems in connection with entropy criteria and showed that such systems are endowed with large families of entropies. From a different perspective, the same type of systems were studied by Tsarev [**Ts**]. Sévennec [**Sev**] showed that the eigenvalues of strictly hyperbolic, rich systems must satisfy restrictive symmetry conditions. Serre [**Ser**] has performed a comprehensive analysis of rich systems, including building of entropies, commuting families of rich systems, and construction of general rich systems starting from certain structure coefficients. In Section 4.3 below we analyze a sub-class of rich systems in relation to Problem 1. It turns out that there are many (in particular, many strictly hyperbolic) solutions to Problem 1 in this case.

3. Formulating the λ-system

The condition that the matrix $A(u)$ in Problem 1 is a Jacobian matrix with respect to the u coordinates may be formulated in different ways. The various PDE systems will be referred to as *λ-systems*. The most direct approach is to require that

$$\partial_k A^i_j(u) = \partial_j A^i_k(u) \qquad \text{for all } i,j,k = 1,\dots,n \text{ with } j<k\,. \tag{3.1}$$

Here ∂_i denotes partial differentiation with respect to u_i. When written out (3.1) yields a homogeneous system of $\frac{n^2(n-1)}{2}$ linear PDEs for n unknowns. For $n \geq 3$ it is thus typically an overdetermined system of PDEs. However, this formulation is not well-suited for further analysis.

A simpler formulation of the λ-system is obtained by observing that on a contractible domain Ω

$$A(u) \text{ is a Jacobian matrix} \qquad \Longleftrightarrow \qquad dA(u)\wedge du = 0\,, \tag{3.2}$$

where the d-operator is applied component-wise. Applying the product rule, condition (3.2) is thus equivalent to

$$(d\Lambda)\wedge(Ldu) = \big\{\Lambda(LdR) - (LdR)\Lambda\big\}\wedge(Ldu)\,, \tag{3.3}$$

where we have used that $L = R^{-1}$. The system (3.3) is an equation for n-vectors of 2-forms. We proceed to write out the system in u-coordinates by applying each side to the pair of vector fields $(\partial/\partial u^i, \partial/\partial u^j)$. A direct calculation yields:

$$L^i_j(\partial_l\lambda^i) - L^i_l(\partial_j\lambda^i) = \sum_{m\neq i} B^{mi}_{lj}(\lambda^m-\lambda^i)\,, \qquad \forall\, 1\leq i\leq n\,,\ 1\leq l<j\leq n\,, \tag{3.4}$$

where

$$B^{mi}_{lj} := R_m\cdot\Big\{L^m_j(\partial_l L^i) - L^m_l(\partial_j L^i)\Big\}\,. \tag{3.5}$$

Not surprisingly this is again a homogeneous system of $\frac{n^2(n-1)}{2}$ linear PDEs for n unknowns.

It turns out that there is another formulation of the λ-system which shows that the systems (3.1) and (3.4) contain (non-obvious) *algebraic* constraints on the eigenvalues λ^i. To formulate these we let $r_i(u)$, $\ell^i(u)$ denote the vector-fields (i.e. differential operators) and differential 1-forms given by

$$r_i(u) := \sum_{m=1}^n R^m_i\frac{\partial}{\partial u^m}\,, \qquad \ell^i(u) := \sum_{m=1}^n L^i_m du^m\,. \tag{3.6}$$

Since $R(u)$ is assumed invertible on Ω, the vectors $r_i(u)$ provide a frame on Ω, and since $L = R^{-1}$ the forms $\ell^i(u)$ provide the dual coframe on Ω. We set

$$\ell := \begin{bmatrix}\ell^1\\ \vdots\\ \ell^n\end{bmatrix} = Ldu\,,$$

and introduce the following coefficients

$$\Gamma^k_{ij} := L^k(D_u R_j)R_i = \big(Lr_i(R)\big)^k_j\,. \tag{3.7}$$

A direct computation shows that the Γ^k_{ij} are, in fact, the Christoffel symbols of the the standard affine connection $\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j} = 0$ computed relative to the $r_1,\dots,r_n$ frame. Furthermore, the (k,j)-entry of the matrix $\mu := R^{-1}dR = LdR$ of 1-forms is given by

$$\big(LdR\big)^k_j = \sum_{i=1}^n \Gamma^k_{ij}\ell^i = \mu^k_j\,, \tag{3.8}$$

where μ^k_i are the connection forms. Thus (3.3) reads

$$(d\Lambda)\wedge\ell = \big\{\Lambda\mu - \mu\Lambda\big\}\wedge\ell \tag{3.9}$$

This last equation is again an equation between n-vectors of two-forms. By evaluating each component on pairs of frame vector-fields (r_i, r_j), $i, j = 1\dots n$, we obtain an equivalent differential-algebraic system:

$$r_i(\lambda^j) \;=\; \Gamma^j_{ji}(\lambda^i - \lambda^j) \qquad \text{for } i \neq j, \tag{3.10}$$

$$(\lambda^i - \lambda^k)\Gamma^k_{ji} \;=\; (\lambda^j - \lambda^k)\Gamma^k_{ij} \qquad \text{for } i < j,\ i\neq k,\ j\neq k, \tag{3.11}$$

where there are no summations. Here (3.10) gives $n(n-1)$ linear, homogeneous PDEs, while (3.11) gives $\frac{n(n-1)(n-2)}{2}$ linear algebraic relations on λ's.

4. Analyzing the differential-algebraic λ-system

Among the various formulations, it is the differential-algebraic form (3.10)-(3.11) which is best suited for concrete calculations as well as for analyzing how large the set of solutions is.

4.1. Trivial solutions. Even without writing down any equations, it is clear that Problem 1 always has a one-parameter family of trivial solutions given by

$$\lambda^1(u) = \cdots = \lambda^n(u) \equiv \hat\lambda\,,$$

where $\hat\lambda$ is any real constant. The resulting system (1.1) is linearly degenerate in all families and any map $f(u) = \hat\lambda u + \hat u$, where $\hat u \in \mathbb{R}^n$, is a corresponding flux. As shown by Example 5.2 there are cases in which there are *no nontrivial* solutions to the λ-system. For later reference we record the following related result whose proof is immediate.

Proposition 1. *If $\lambda^1(u) = \cdots = \lambda^n(u)$ is a solution to Problem 1, then their common value is necessarily a constant.*

4.2. Algebraic system; extreme cases. We proceed to analyze the algebraic constraints (3.11) which, on its own, constitute a linear set of $\frac{n(n-1)(n-2)}{2}$ equations for $n-1$ unknowns. We may choose the unknowns to be the differences

$$x^k := \lambda^k - \lambda^1\,,$$

and write (3.11) as

$$Nx = 0\,, \tag{4.1}$$

where x is the $(n-1)$-vector $(x^2, \ldots, x^n)^T$ and N is a certain $\frac{n(n-1)(n-2)}{2} \times (n-1)$-matrix whose entries are given in terms of the Γ^k_{ij}. The number of independent algebraic conditions depends on $\operatorname{rank} N$ and it is natural to use this rank as a first, rough classification. Let us record the two extreme cases:

(i) $\operatorname{rank} N = 0$. In this case $N \equiv 0$ and there are no algebraic constraints imposed on the eigenvalues. It can be shown that this occurs if and only if $\Gamma^k_{ij} = 0$ for all choices of $i \neq j \neq k \neq i$. Since the symmetry of the standard connection (see (2.3) and (2.12)) implies that $[r_i, r_j] = \sum_{k=1}^n (\Gamma^k_{ij} - \Gamma^k_{ji}) r_k$, it follows that $[r_i, r_j] \in \operatorname{span}\{r_i, r_j\}$ $\forall\, i,\, j$, which is equivalent to the system being rich. Thus $\operatorname{rank} N = 0$ implies richness, and in the next subsection it is shown that Problem 1 has a large solution set whenever $\operatorname{rank} N = 0$. (We stress however that richness of the prescribed eigenfields, i.e. $[r_i, r_j] \in \operatorname{span}\{r_i, r_j\}$ $\forall\, i,\, j$, does *not* imply $\operatorname{rank} N = 0$; see Example 5.3).

(ii) $\operatorname{rank} N = n - 1$. This is the case if and only if the only solution to (4.1) is $x = 0$, that is, all λ^i are equal. According to Proposition 1 this is the case if and only if all eigenvalues are equal to a common *constant*. Thus, in this case the λ-system admits only trivial solutions, i.e. the general solution depends on one constant.

It will be clear from the following analysis of the case of $n = 3$ that the intermediate cases where $1 \leq \operatorname{rank} N \leq n - 2$ are more involved. In particular, knowing $\operatorname{rank} N$ is *not* enough to characterize the size of the set of solutions to the λ-system.

4.3. The solution set when $\operatorname{rank} N = 0$**.** We first note that the case of two equations ($n = 2$) falls into this class since there are no algebraic constraints in this case. As observed above, $\operatorname{rank} N = 0$ implies that $[R_i, R_j] \in \operatorname{span}\{R_i, R_j\}$ $\forall\, i,\, j$, and in this case there exists a set of (Riemann) coordinates $w = (w^1, \ldots, w^n)$ whose gradients are linearly independent and satisfy

$$\nabla w^i(u) \cdot R_j(u) \begin{cases} = 0 & \text{if } i \neq j\,, \\ \neq 0 & \text{if } i = j\,. \end{cases}$$

See [**D2**] and [**Ser**] for details. Equivalently there exist scalar functions $\alpha_i(u)$, $i = 1, \ldots, n$ such that $\frac{\partial}{\partial w^i} = \alpha_i(u)\, r_i$, $i = 1, \ldots, n$. In the w-coordinates the system (3.10) is equivalent to

$$\frac{\partial}{\partial w_i}(k^j) = Z^j_{ji}(k^i - k^j) \qquad \text{for } i \neq j, \tag{4.2}$$

where $k^i(w) = \lambda^i(u(w))$, $i = 1, \ldots, n$, are the unknown eigenvalues in the w-coordinates, and $Z^k_{ij}(w)$ are the Christoffel symbols corresponding to the connection (3.7) relative to the frame $\{\frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^n}\}$. From the symmetry of the connection (2.12) follows the lower-indices symmetry $Z^k_{ij}(w) = Z^k_{ji}(w)$. From the flatness condition (2.13) one can deduce that (4.2) is compatible [**JK**]. Moreover, it can be proved, using either the Cartan-Kähler theorem (Theorem 7.3.3. [**IL**]) or a result of Darboux (Livre III, Chapitre I, Théorème III, [**Dar**]), that the general solution of (4.2) depends on n arbitrary functions of one variable and at most n arbitrary constants. In Example 5.1 we describe the set of solutions of rich Euler systems, which are $\operatorname{rank} N = 0$ systems.

4.4. The case $n = 3$. The λ-system is now an overdetermined system of 9 linear equations for the 3 unknowns $\lambda^1(u)$, $\lambda^2(u)$, $\lambda^3(u)$. In this case there are three possibilities for rank N, and the previous analysis dealt with the cases rank $N = 0$ (general solution depends on three functions of one variable and and at most three constants) and rank $N = 2$ (general solution is $\lambda^1 = \lambda^2 = \lambda^3 = const$). On the other hand, the case rank $N = 1$ requires further subdivisions. Let us write (in this case) the unique, nontrivial algebraic relation implied by the λ-system as

$$\alpha_1\lambda^1 + \alpha_2\lambda^2 + \alpha_3\lambda^3 = 0. \tag{4.3}$$

It turns out that $\alpha_3 = -(\alpha_1 + \alpha_2)$ such that there are two further possibilities:

(1) all three λ's are involved in (4.3) with non-zero coefficients
(2) only two of the three λ's are involved in (4.3) with non-zero coefficients.

In [**JK**] it is shown that in the Case (1) the system is equivalent to a Frobenius system with two unknown functions of three variables ("Frobenius" means that the derivatives in all three eigen-directions of both unknowns are prescribed). Thus, provided the compatibility conditions for this system are satisfied the general solution will depend on two constants. In Example 5.1 we describe the set of solutions of non-rich Euler systems, that falls into this category. With Example 5.2 we show that the compatibility conditions may fail, in general, in which case the only solution is trivial $\lambda^1 = \lambda^2 = \lambda^3 = const$. We prove in [**JK**] that there are no rich systems in rank $N = 1$ Case (1) category.

In Case (2), there may be only trivial solutions, or, otherwise, using Cartan-Kaähler theorem, we prove in [**JK**] that the general solution depends on one arbitrary function of one variable and at most two constants. There are both rich (see Example 5.3) and non-rich (see Example 5.4) systems with rank $N = 1$ in the Case (2) category.

5. Examples

Example 5.1. Consider the Euler equations for 1-dimensional compressible gas flow written in Lagrangian variables in the form

$$v_t - u_x = 0 \tag{5.1}$$

$$u_t + p_x = 0 \tag{5.2}$$

$$S_t = 0\,, \tag{5.3}$$

where v, u, S, p are specific volume, velocity, specific entropy, and pressure, respectively. We assume that the pressure is a prescribed function $p = p(v, S)$ with $p_v(v, S) < 0$. In this case the eigenvectors of the Jacobian of the flux $(-u, p, 0)^T$ are given solely in terms of the pressure function:

$$R_1 = (1,\, \sqrt{-p_v}\,, 0)^T, \qquad R_2 = (-p_S,\, 0,\, p_v)^T, \qquad R_3 = (1,\, -\sqrt{-p_v},\, 0)^T, \tag{5.4}$$

while the eigenvalues are given by

$$\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}\,. \tag{5.5}$$

We now pose the inverse problem: how many systems of three conservation laws (in the variables (v, u, S)) are there, with the same set of eigenvectors (5.4)? That is, we want to know the set of solutions to the λ-system corresponding to (5.4). In [**JK**] a complete answer to this question is given as follows:

RICH GAS DYNAMICS: i.e. the vector fields R_1, R_2, R_3 are pairwise in involution. This occurs if and only if $p = p(v,S)$ satisfies $\left(\frac{p_S}{p_v}\right)_v \equiv 0$. The pressure must then be of the form $p(v,S) = \Phi(v + \psi(S))$ (Φ and ψ given), and the general solution is of the form: λ^2 is any function of S alone, while λ^1 and λ^3 are functions of $v + \psi(S)$ and u. To determine λ^1 and λ^3 requires two arbitrary functions of one variable. All in all, the solution in this case depends on three functions of one variable.

NON-RICH GAS DYNAMICS:$\left(\frac{p_S}{p_v}\right)_v \neq 0$. In this case the λ-system implies a single algebraic constraint that involves all three λ's:

$$2\lambda^2 = \lambda^1 + \lambda^3\,, \tag{5.6}$$

and the solution of the λ-system is parametrized by two constants $\bar{\lambda}$ and C as follows:

$$\lambda^1 = \bar{\lambda} - C\sqrt{-p_v}, \quad \lambda^2 \equiv \bar{\lambda}, \quad \lambda^3 = \bar{\lambda} + C\sqrt{-p_v}\,. \tag{5.7}$$

EXAMPLE 5.2. Consider the right eigenvectors:

$$R_1 = (0,\,0,\,1)^T, \qquad R_2 = (0,\,1,\,u^1)^T \qquad R_3 = (u^3,\,0,\,1)^T.$$

The PDEs in the λ-system are given by

$$\begin{aligned}
r_1(\lambda^2) &= \partial_3\lambda^2 = 0,\\
r_1(\lambda^3) &= \partial_3\lambda^3 = 0,\\
r_2(\lambda^1) &= \partial_2\lambda^1 + u^1\partial_3\lambda^1 = 0,\\
r_2(\lambda^3) &= \partial_2\lambda^3 + u^1\partial_3\lambda^3 = 0,\\
r_3(\lambda^1) &= u^3\partial_1\lambda^1 + \partial_3\lambda^1 + \frac{1}{u}^3(\lambda^3 - \lambda^1) = 0,\\
r_3(\lambda^2) &= u^3\partial_1\lambda^2 + \partial_3\lambda^2 = 0,
\end{aligned}$$

while there is single algebraic relation that involves all three λ's:

$$(u^3)^2\,(\lambda^2 - \lambda^1) + u^1\,(\lambda^3 - \lambda^1) = 0.$$

The only solution of the above system of differential and algebraic conditions is trivial $\lambda^1 = \lambda^2 = \lambda^3 = const.$

EXAMPLE 5.3. Consider a system with eigenvectors:

$$R_1 = (1,0,u^2)^T, \quad R_2 = (0,1,u^1)^T, \quad R_3 = (0,0,-1)^T.$$

These vectors commute and therefore any system (1.1) with these as eigenvectors is rich. The only non-zero connection components are $\Gamma^3_{12} = \Gamma^3_{21} = -1$, such that the λ-system implies a single algebraic relation: $\lambda^1 = \lambda^2$. Thus this is an example of a rich, rank $N = 1$, Case(2) system. Taking into account this relation the differential equations become:

$$\begin{aligned}
r_1(\lambda^1) &= \partial_1\lambda^1 + u^2\partial_3\lambda^1 = 0,\\
r_1(\lambda^3) &= \partial_1\lambda^3 + u^2\partial_3\lambda^3 = 0,\\
r_2(\lambda^1) &= \partial_2\lambda^1 + u^1\partial_3\lambda^1 = 0,\\
r_2(\lambda^3) &= \partial_2\lambda^3 + u^1\partial_3\lambda^3 = 0,\\
r_3(\lambda^1) &= -\partial_3\lambda^1 = 0.
\end{aligned}$$

The general solution of this system is $\lambda^1 = \lambda^2 \equiv \bar{\lambda}$, and $\lambda^3 = \varphi(u^3 - u^1 u^2)$, where $\bar{\lambda}$ is an arbitrary constant and φ is an arbitrary function of one variable.

Example 5.4. Consider a system with eigenvectors:

$$R_1 = (1, 0, u^2)^T, \quad R_2 = (0, 1, 0)^T, \quad R_3 = (u^2, 0, -1)^T.$$

The commutator relations are $[r_1, r_2] = \frac{1}{1+(u^2)^2}(r_3 - u^2\, r_1)$, $[r_1, r_3] = 0$, and $[r_2, r_3] = \frac{1}{1+(u^2)^2}\,(r_1 + u^2\, r_3)$. These vector-fields are not pair-wise in involution, and therefore the system is not rich. The λ-system implies a unique algebraic relations $\lambda^1 = \lambda^3$, which involves only two of λ's. Thus this is an example of a non-rich, rank $N = 1$, Case(2) system. Taking into account this relation the differential equations become:

$$\begin{aligned}
r_1(\lambda^2) &= \partial_1 \lambda^2 + u^2\, \partial_3 \lambda^2 = 0,\\
r_1(\lambda^3) &= \partial_1 \lambda^3 + u^2\, \partial_3 \lambda^3 = 0,\\
r_2(\lambda^3) &= \partial_2 \lambda^3 = 0,\\
r_3(\lambda^3) &= u^2\, \partial_1 \lambda^3 - \partial_3 \lambda^3 = 0,\\
r_3(\lambda^2) &= u^2\, \partial_1 \lambda^2 - \partial_3 \lambda^2 = 0.
\end{aligned}$$

The general solution of this system is $\lambda^1 = \lambda^3 \equiv \bar{\lambda}$, $\lambda^2 = \varphi(u^2)$, where, again, $\bar{\lambda}$ is an arbitrary constant and φ is an arbitrary function of one variable.

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