# Object-Image Correspondence for Curves under Central and Parallel Projections 

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#### Abstract

We present a novel algorithm for deciding whether a given planar curve is an image of a given spatial curve, obtained by a central or a parallel projection with unknown parameters. A straightforward approach to this problem consists of setting up a system of conditions on the projection parameters and then checking whether or not this system has a solution. The computational advantage of the algorithm presented here, in comparison to algorithms based on the straightforward approach, lies in a significant reduction of a number of real parameters that need to be eliminated in order to establish existence or non-existence of a projection that maps a given spatial curve to a given planar curve. Our algorithm is based on projection criteria that reduce the projection problem to a certain modification of the equivalence problem of planar curves under affine and projective transformations. The latter problem is then solved using a separating set of rational differential invariants. A similar approach can be used to decide whether a given finite list of points on a plane is an image of a given finite list of points in $\mathbb{R}^{3}$. The motivation comes from the problem of establishing a correspondence between an object and an image, taken by a camera with unknown position and parameters.


## Categories and Subject Descriptors

I.1.4 [Symbolic and Algebraic Manipulations]: Applications

## General Terms

Algorithms, Theory

[^0]Figure 1: A pinhole camera [24].

## Keywords

Central and parallel projections; finite and affine cameras; curves; projective and affine transformations; separating differential invariants; signatures; machine vision.

## 1. INTRODUCTION

A central projection from $\mathbb{R}^{3}$ to $\mathbb{R}^{2}$ models a simple pinhole camera pictured in Figure 1. A generic central projection is described by a linear fractional transformation (1):

$$
\begin{align*}
& x=\frac{p_{11} z_{1}+p_{12} z_{2}+p_{13} z_{3}+p_{14}}{p_{31} z_{1}+p_{32} z_{2}+p_{33} z_{3}+p_{34}}, \\
& y=\frac{p_{21} z_{1}+p_{22} z_{2}+p_{23} z_{3}+p_{24}}{p_{31} z_{1}+p_{32} z_{2}+p_{33} z_{3}+p_{34}}, \tag{1}
\end{align*}
$$

where $\left(z_{1}, z_{2}, z_{3}\right)$ denote coordinates in $\mathbb{R}^{3},(x, y)$ denote coordinates in $\mathbb{R}^{2}$ and $p_{i j}, i=1 \ldots 3, j=1 \ldots 4$, are real parameters of the projection, such that the left $3 \times 3$ submatrix of $3 \times 4$ matrix $P=\left(p_{i j}\right)$ has a non-zero determinant. Parameters represent the freedom to choose the center of the projection, the position of the image plane and (in general non-orthogonal) coordinate system on the image plane. ${ }^{1}$ In the case when the distance between a camera and an object is significantly greater than the object depth, a parallel projection provides a good camera model. A parallel projection has 8 parameters and can be described by a $3 \times 4$ matrix of rank 3 , whose last row is $(0,0,0,1)$. We review various camera models and related geometry in Section 2 (see also [17]). In most general terms, the projection problem considered here is formulated as follows:

[^1]Problem 1. Given a subset $\mathcal{Z}$ of $\mathbb{R}^{3}$ and a subset $\mathcal{X}$ of $\mathbb{R}^{2}$, does there exist a projection $P: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ such that $\mathcal{X}=$ $P(\mathcal{Z})$ ?

A straightforward approach to this problem consists of setting up a system of conditions on the projection parameters and then checking whether or not this system has a solution. In the case when $\mathcal{Z}$ and $\mathcal{X}$ are finite lists of points, a solution based on the straightforward approach can be found in [17]. For curves and surfaces under central projections, this approach is taken in [12]. However, internal parameters of the camera are considered to be known in that paper and, therefore, there are only 6 camera parameters in that study vs. 12 considered here. The method presented in [12] also uses an additional assumption that a planar curve $\mathcal{X} \subset \mathbb{R}^{2}$ has at least two points, whose tangent lines coincide. An alternative approach to the problem in the case when $\mathcal{Z}$ and $\mathcal{X}$ are finite lists of points under parallel projections was presented in $[2,1]$. In these articles, the authors establish polynomial relationships that have to be satisfied by coordinates of the points in the sets $\mathcal{Z}$ and $\mathcal{X}$ in order for a projection to exists. Our approach to the projection problem for curves is closer in spirit to $[2,1]$, as we also establish necessary and sufficient conditions that curves $\mathcal{Z}$ and $\mathcal{X}$ must satisfy in order for a projection to exist. However, unlike [2, 1], we exploit the relationship between the projection problem and equivalence problem under group-actions. We will show below that, in comparison with the straightforward approach, our solution leads to a significant reduction of the number of parameters that have to be eliminated in order to solve Problem 1 for curves.

In this paper, we assume that $\mathcal{Z}$ and $\mathcal{X}$ are rational algebraic curves, i. e. $\mathcal{Z}=\{\Gamma(s) \mid s \in \mathbb{R}\}$ and $\mathcal{X}=\{\gamma(t) \mid t \in \mathbb{R}\}$, where $\Gamma: \mathbb{R} \rightarrow \mathbb{R}^{3}$ and $\gamma: \mathbb{R} \rightarrow \mathbb{R}^{2}$ are rational maps, and that $\mathcal{Z}$ is not a straight line (and, therefore, its image under any projection is a one-dimensional constructible set). We also relax the projection condition to $\mathcal{X}=\overline{P(\mathcal{Z})}$, where bar throughout the paper denotes the algebraic closure of a set. The case of non-rational algebraic curves in more technical and is discussed in [6]. Then Problem 1, for central projections, can be reformulated as the following real quantifier elimination problem:

Reformulation 1. (STRAIGHTFORWARD APPROACH) Given two rational maps $\Gamma: \mathbb{R} \rightarrow \mathbb{R}^{3}$ and $\gamma: \mathbb{R} \rightarrow \mathbb{R}^{2}$, determine the truth of the statement:

$$
\begin{aligned}
& \exists P \in \mathbb{R}^{3 \times 4} \quad \operatorname{det}\left(p_{i j}\right)_{i=1 \ldots 3}^{j=1 \ldots 3} \neq 0 \\
& \forall s \text { in the domain of } \Gamma(s) \quad \exists t \in \mathbb{R} \quad \gamma(t)=P(\Gamma(s)) .
\end{aligned}
$$

Real quantifier elimination problems are algorithmically solvable [22]. A survey of subsequent developments in this area can be found, for instance, in [18] and [9]. High computational complexity of these problems makes a reduction in the number of parameters to be desirable.

The projection criteria developed in this paper reduces the projection problem to the problem of deciding whether the given planar curve $\mathcal{X}$ is equivalent to a curve in a certain family of planar curves under an action of the projective group in the case of central projections, and under the action of the affine group in the case of parallel projections. The family of curves depends on 3 parameters in the case of central projections, and on 2 parameters in the case of parallel projections.

These group-equivalence problems can be solved by an adaptation of differential signature construction developed in [7] to solve local equivalence problems for smooth curves. In Section 4, we give an algebraic formulation of signature construction and show that it leads to a solution of global equivalence problems for algebraic curves. For this purpose we introduce a notion of differentially separating set of invariants. Following this method for the case of central projections, when $\mathcal{Z}$ and $\mathcal{X}$ are rational algebraic curves, we define two rational signature maps $\mathcal{S}_{\mathcal{X}}: \mathbb{R} \rightarrow \mathbb{R}^{2}$ and $\mathcal{S}_{\mathcal{Z}}: \mathbb{R}^{4} \rightarrow \mathbb{R}^{2}$. Construction of these signature maps requires only differentiation and arithmetic operations and is computationally trivial. Problem 1 becomes equivalent to

Reformulation 2. (SIGNature approach) Given two rational maps $S_{\mathcal{X}}: \mathbb{R} \rightarrow \mathbb{R}^{2}$ and $S_{\mathcal{Z}}: \mathbb{R}^{4} \rightarrow \mathbb{R}^{2}$, determine the truth of the statement:

$$
\begin{aligned}
& \exists c \in \mathbb{R}^{3} \quad \forall s \text { in the domain of } S_{\mathcal{Z}}(c, s) \\
& \exists t \in \mathbb{R} \quad S_{\mathcal{Z}}(c, s)=S_{\mathcal{X}}(t) .
\end{aligned}
$$

We note that Reformulation 1 and Reformulation 2 have similar structure, but the former requires elimination of 14 parameters, while the latter requires elimination of only 5 parameters. ${ }^{2}$ The case of parallel projection is treated in the similar manner and leads to the reduction of the number of real parameters that need to be eliminated from 10 to 4 .

Our method can be easily adapted to solve projection problems for curves in $\mathbb{R}^{n}$ projected to a hyperplane. The problem and the solution remains valid if we replace $\mathbb{R}$ with $\mathbb{C}$ and, in fact, as with most problems in algebraic geometry, the implementation of the algorithms is easier over the complex numbers.. The existence of a complex projection provides a necessary but not a sufficient condition for the existence of a real projection. In Section 6, we discuss how this method leads to an alternative solution to the projection problem for finite lists of points (see [5] for more details).

Although the relation between projections and group actions is known, our literature search did not yield algorithms that exploit this relationship to solve the projection problem for curves in the generic setting of cameras with unknown internal and external parameters. The goal of the paper is to introduce such algorithms. Although the development of efficient implementations of these algorithms and their complexity study lie outside of the scope of this paper, we made a preliminary implementations in Maple of projection algorithms over complex numbers. We implemented both an algorithm based on signature construction presented here and an algorithm based on the straightforward approach and timed their performance. The code and the experiments are posted on the internet [25].

In order to become practically useful in real-life applications, the algorithmic solution presented here, would have to be adapted to curves given by a finite sampling of points. Some directions of such adaptation are indicated in Section 7 of the paper.

## 2. PROJECTIONS AND CAMERAS

We embed $\mathbb{R}^{n}$ into projective space $\mathbb{P}^{n}$ and use homogeneous coordinates on $\mathbb{P R}^{n}$ to express the map (1) by matrix multiplication.

[^2]Notation 2. Square brackets around matrices (and, in particular, vectors) will be used to denote an equivalence class with respect to multiplication of a matrix by a nonzero scalar. Multiplication of equivalence classes of matrices $A$ and $B$ of appropriate sizes is well-defined by $[A][B]:=[A B]$.

With this notation, a point $(x, y) \in \mathbb{R}^{2}$ corresponds to a point $[x, y, 1]=[\lambda x, \lambda y, \lambda] \in \mathbb{P}^{2}$ for all $\lambda \neq 0$, and a point $\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{R}^{3}$ corresponds to $\left[z_{1}, z_{2}, z_{3}, 1\right] \in \mathbb{P R}^{3}$. We will refer to the points in $\mathbb{P R}^{n}$ whose last homogeneous coordinate is zero as points at infinity. In homogeneous coordinates projection (1) is a map $[P]: \mathbb{P R}^{3} \rightarrow \mathbb{P R}^{2}$ given by

$$
\begin{equation*}
[x, y, 1]^{\mathrm{T}}=[P]\left[z_{1}, z_{2}, z_{3}, 1\right]^{\mathrm{T}} \tag{2}
\end{equation*}
$$

where $P$ is $3 \times 4$ matrix of rank 3 and superscript T denotes transposition. Matrix $P$ has a 1-dimensional kernel. Therefore, there exists a point $\left[z_{1}^{0}, z_{2}^{0}, z_{3}^{0}, z_{4}^{0}\right] \in \mathbb{P R}^{3}$ whose image under the projection is undefined (recall that $[0,0,0]$ is not a point in $\left.\mathbb{P R}^{2}\right)$. Geometrically, this point is the center of the projection.

In computer science literature (e.g. [17]), a camera is called finite if its center is not at infinity. A finite camera is modeled by a matrix $P$, whose left $3 \times 3$ submatrix is nonsingular. Geometrically, finite cameras correspond to central projections from $\mathbb{R}^{3}$ to a plane. On the contrary, an infinite camera has its center at an infinite point of $\mathbb{P R}^{3}$. An infinite camera is modeled by a matrix $P$ whose left $3 \times 3$ submatrix is singular. An infinite camera is called affine if the preimage of the line at infinity in $\mathbb{P R}^{2}$ is the plane at infinity in $\mathbb{P R}^{3}$. An affine camera is modeled by a matrix $P$ whose last row is $(0,0,0,1)$. In this case map (1) becomes $x=$ $p_{11} z_{1}+p_{12} z_{2}+p_{13} z_{3}+p_{14}, \quad y=p_{21} z_{1}+p_{22} z_{2}+p_{23} z_{3}+p_{24}$. Geometrically, affine cameras correspond to parallel projections from $\mathbb{R}^{3}$ to a plane. ${ }^{3}$ Eight degrees of freedom reflect a choice of the direction of a projection, a position of the image plane and a choice of linear system of coordinates on the image plane. An image plane may be assumed to be perpendicular to the direction of the projection, since other choices are absorbed in the freedom to choose, in general, non-orthogonal coordinate system on the image plane.

Definition 3. A set of equivalence classes $[P]$, where $P=\left(p_{i j}\right)_{j=1 \ldots 4}^{i=1 \ldots 3}$ is a $3 \times 4$ matrix whose left $3 \times 3$ submatrix is non-singular, is called the set of central projections and is denoted $\mathcal{C P}$.

A set of equivalence classes $[P]$, where $P=\left(p_{i j}\right)_{j=1 \ldots 4}^{i=1 \ldots . .4}$ has rank 3 and its last row is $(0,0,0, \lambda), \lambda \neq 0$, is called the set of parallel projections and is denoted $\mathcal{P P}$.

Equation (1) determines a central projection when $[P] \in \mathcal{C P}$ and it determines a parallel projection when $[P] \in \mathcal{P} \mathcal{P}$. Sets $\mathcal{C P}$ and $\mathcal{P} \mathcal{P}$ are disjoint. Projections that are not included in these two classes correspond to infinite, non-affine cameras. These are not frequently used in computer vision and are not considered in this paper.

## 3. PROJECTION CRITERIA FOR CURVES

Recall that for every algebraic curve $\mathcal{X} \subset \mathbb{R}^{n}$ there exists a unique projective algebraic curve $[\mathcal{X}] \subset \mathbb{P R}^{n}$ such that $[\mathcal{X}]$ is the smallest projective variety containing $\mathcal{X}$ (see [14]).

[^3]Definition 4. We say that a curve $\mathcal{Z} \subset \mathbb{R}^{3}$ projects to $\mathcal{X} \subset \mathbb{R}^{2}$ if there exists a $3 \times 4$ matrix $P$ of rank 3 such that $[\mathcal{X}]=\overline{\{[P][\mathbf{z}] \mid \mathbf{z} \in \mathcal{Z}, P \mathbf{z} \neq 0\}}$. We then write $\mathcal{X}=P(\mathcal{Z})$ or $[\mathcal{X}]=[P][\mathcal{Z}]$.

Definition 5. The projective group $\mathcal{P G \mathcal { L }}(n+1)$ is a quotient of the general linear group $\mathcal{G} \mathcal{L}(n+1)$, consisting of $(n+1) \times(n+1)$ non-singular matrices, by a 1-dimensional abelian subgroup $\lambda I$, where $\lambda \neq 0 \in \mathbb{R}$ and $I$ is the identity matrix. Elements of $\mathcal{P G \mathcal { L }}(n+1)$ are equivalence classes $[B]=[\lambda B]$, where $\lambda \neq 0$ and $B \in \mathcal{G} \mathcal{L}(n+1)$.

The affine group $\mathcal{A}(n)$ is a subgroup of $\mathcal{P G} \mathcal{L}(n+1)$ whose elements $[B]$ have a representative $B \in \mathcal{G} \mathcal{L}(n+1)$ with the last row equal to $(0, \ldots, 0,1)$.

The special affine group $\mathcal{S A}(n)$ is a subgroup of $\mathcal{A}(n)$ whose elements $[B]$ have a representative $B \in \mathcal{G} \mathcal{L}(n+1)$ with determinant 1 and the last row equal to $(0, \ldots, 0,1)$.

In homogeneous coordinates, the standard action of the projective group $\mathcal{P G} \mathcal{L}(n+1)$ on $\mathbb{P R}^{n}$ is defined by multiplication:

$$
\begin{equation*}
\left[z_{1}, \ldots, z_{n}, z_{0}\right]^{\mathrm{T}} \rightarrow[B]\left[z_{1}, \ldots, z_{n}, z_{0}\right]^{\mathrm{T}} \tag{3}
\end{equation*}
$$

The action (3) induces linear-fractional action of $\mathcal{P G \mathcal { L }}(n+1)$ on $\mathbb{R}^{n}$. The restriction of (3) to $\mathcal{A}(n)$ induces an action on $\mathbb{R}^{n}$ consisting of compositions of linear transformations and translations.

Definition 6. We say that two curves $\mathcal{X}_{1} \subset \mathbb{R}^{n}$ and $\mathcal{X}_{2} \subset \mathbb{R}^{n}$ are $\mathcal{P G} \mathcal{L}(n+1)$-equivalent if there exists $[A] \in$ $\mathcal{P G \mathcal { L }}(n+1)$, such that $\left[\mathcal{X}_{2}\right]=\left\{[A][\mathbf{p}] \mid[\mathbf{p}] \in\left[\mathcal{X}_{1}\right]\right\}$. We then write $\mathcal{X}_{2}=A\left(\mathcal{X}_{1}\right)$ or $\left[\mathcal{X}_{2}\right]=[A]\left[\mathcal{X}_{1}\right]$. If $[A] \in G$, where $G$ is a subgroup of $\mathcal{P G} \mathcal{L}(n+1)$, we say that $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are $G$-equivalent.

Before stating the projection criteria, we make the following simple, but important observations.

Proposition 7. (i) If $\mathcal{Z} \subset \mathbb{R}^{3}$ projects to $\mathcal{X} \subset \mathbb{R}^{2}$ by a parallel projection, then any curve that is $\mathcal{A}(3)-$ equivalent to $\mathcal{Z}$ projects to any curve that is $\mathcal{A}(2)$ equivalent to $\mathcal{X}$ by a parallel projection. In other words, parallel projections are defined on affine equivalence classes of curves.
(ii) If $\mathcal{Z} \subset \mathbb{R}^{3}$ projects to $\mathcal{X} \subset \mathbb{R}^{2}$ by a central projection then any curve in $\mathbb{R}^{3}$ that is $\mathcal{A}(3)$-equivalent to $\mathcal{Z}$ projects to any curve on $\mathbb{R}^{2}$ that is $\mathcal{P G \mathcal { L }}(3)$-equivalent to $\mathcal{X}$ by a central projection.
Proof. (i) Assume that there exists a parallel projection $[P] \in \mathcal{P} \mathcal{P}$ such that $[\mathcal{X}]=[P][\mathcal{Z}]$. Then for all $(A, B) \in$ $\mathcal{A}(2) \times \mathcal{A}(3)$ we have $[A][\mathcal{X}]=[A][P]\left[B^{-1}\right][B][\mathcal{Z}]$. Since $[A][P]\left[B^{-1}\right] \in \mathcal{P} \mathcal{P}$, curve $B(\mathcal{Z})$ projects to $A(\mathcal{X})$. (ii) is proved similarly.

Remark 8. It is known that if $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are images of a curve $\mathcal{Z}$ under two central projections with the same center, then $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are $\mathcal{P G \mathcal { L }}(3)$-equivalent, but if the centers of the projections are not the same this is no longer true (see Example 33). Similarly, images of $\mathcal{Z}$ under different parallel projections may not be $\mathcal{A}(2)$-equivalent (see Example 35).

Theorem 9. (Central projection criterion.) A curve $\mathcal{Z} \subset \mathbb{R}^{3}$ projects to a curve $\mathcal{X} \subset \mathbb{R}^{2}$ by a central
projection if and only if there exist $c_{1}, c_{2}, c_{3} \in \mathbb{R}$ such that $\mathcal{X}$ is $\mathcal{P G \mathcal { L }}(3)$-equivalent to a planar curve

$$
\begin{equation*}
\tilde{\mathcal{Z}}_{c_{1}, c_{2}, c_{3}}=\overline{\left\{\left(\frac{z_{1}+c_{1}}{z_{3}+c_{3}},\right.\right.} \frac{\left.\left.\frac{z_{2}+c_{2}}{z_{3}+c_{3}}\right) \mid\left(z_{1}, z_{2}, z_{3}\right) \in \mathcal{Z}\right\}}{} \tag{4}
\end{equation*}
$$

Proof. $(\Rightarrow)$ Assume there exists a central projection $[P]$ such that $[\mathcal{X}]=[P][\mathcal{Z}]$. Then $P$ is a $3 \times 4$ matrix $P=$ $\left(p_{i j}\right)_{j=1 \ldots 4}^{i=1 \ldots 3}$ whose left $3 \times 3$ submatrix is non-singular. Therefore there exist $c_{1}, c_{2}, c_{3} \in \mathbb{R}$ such that $p_{* 4}=c_{1} p_{* 1}+c_{2} p_{* 2}+$ $c_{3} p_{* 3}$, where $p_{* j}$ denotes the $j$-th column of the matrix $P$. We observe that

$$
\begin{equation*}
[A]\left[P_{\mathcal{C}}^{0}\right][B]=[P] \tag{5}
\end{equation*}
$$

where $A:=\left(p_{i j}\right)_{j=1 \ldots 3}^{i=1 \ldots 3}$ is $3 \times 3$ submatrix of $P$,

$$
P_{\mathcal{C}}^{0}:=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{6}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right) \text { and } B:=\left(\begin{array}{cccc}
1 & 0 & 0 & c_{1} \\
0 & 1 & 0 & c_{2} \\
0 & 0 & 1 & c_{3} \\
0 & 0 & 0 & 1
\end{array}\right)
$$

Note that $[A]$ belongs to $\mathcal{P G \mathcal { L }}(3)$. Since

$$
\left[P_{\mathcal{C}}^{0}\right][B]\left[z_{1}, z_{2}, z_{3}, 1\right]^{\mathrm{T}}=\left[z_{1}+c_{1}, z_{2}+c_{2}, z_{3}+c_{3}\right]^{\mathrm{T}}
$$

then $[\mathcal{X}]=[A]\left[\tilde{\mathcal{Z}}_{c_{1}, c_{2}, c_{3}}\right]$, where $\tilde{\mathcal{Z}}_{c_{1}, c_{2}, c_{3}}$ is defined by (4).
$(\Leftarrow)$ To prove the converse direction we assume that there exists $[A] \in \mathcal{P G} \mathcal{L}(3)$ and $c_{1}, c_{2}, c_{3} \in \mathbb{R}$ such that $[\mathcal{X}]=$ $[A]\left[\tilde{\mathcal{Z}}_{c_{1}, c_{2}, c_{3}}\right]$, where $\tilde{\mathcal{Z}}_{c_{1}, c_{2}, c_{3}}$ is defined by (4). A direct computation shows that $\mathcal{Z}$ is projected to $\mathcal{X}$ by the central projection $[P]=[A]\left[P_{\mathcal{C}}^{0}\right][B]$, where $B$ and $\left[P_{\mathcal{C}}^{0}\right]$ are given by (6).

Theorem 10. (Parallel projection criterion.) A curve $\mathcal{Z} \subset \mathbb{R}^{3}$ projects to a curve $\mathcal{X} \subset \mathbb{R}^{2}$ by a parallel projection if and only if there exist $c_{1}, c_{2} \in \mathbb{R}$ and an ordered triplet $(i, j, k) \in\{(1,2,3),(1,3,2),(2,3,1)\}$ such that $\mathcal{X}$ is $\mathcal{A}(2)$-equivalent to

$$
\begin{equation*}
\tilde{\mathcal{Z}}_{c_{1}, c_{2}}^{i, j, k}=\overline{\left\{\left(z_{i}+c_{1} z_{k}, \quad z_{j}+c_{2} z_{k}\right) \mid\left(z_{1}, z_{2}, z_{3}\right) \in \mathcal{Z}\right\}} \tag{7}
\end{equation*}
$$

Proof. $(\Rightarrow)$ Assume there exists a parallel projection $[P]$ such that $[\mathcal{X}]=[P][\mathcal{Z}]$. Then $[P]$ can be represented by a matrix:

$$
P=\left(\begin{array}{cccc}
p_{11} & p_{12} & p_{13} & p_{14}  \tag{8}\\
p_{21} & p_{22} & p_{23} & p_{24} \\
0 & 0 & 0 & 1
\end{array}\right)
$$

of rank 3. Therefore there exist $1 \leq i<j \leq 3$ such that the rank of the submatrix $\left(\begin{array}{ll}p_{1 i} & p_{1 j} \\ p_{2 i} & p_{2 j}\end{array}\right)$ is 2 . Then for $1 \leq k \leq$ 3 , such that $k \neq i$ and $k \neq j$, there exist $c_{1}, c_{2} \in \mathbb{R}$, such that $\binom{p_{1 k}}{p_{2 k}}=c_{1}\binom{p_{1 i}}{p_{2 i}}+c_{2}\binom{p_{1 j}}{p_{2 j}}$. We define $A:=$ $\left(\begin{array}{ccc}p_{1 i} & p_{1 j} & p_{14} \\ p_{2 i} & p_{2 j} & p_{24} \\ 0 & 0 & 1\end{array}\right)$ and define $B$ to be the matrix whose columns are vectors $b_{* i}:=(1,0,0,0)^{\mathrm{T}}, b_{* j}:=(0,1,0,0)^{\mathrm{T}}$, $b_{* k}:=\left(c_{1}, c_{2}, 1,0\right)^{\mathrm{T}}, b_{* 4}=(0,0,0,1)^{\mathrm{T}}$. We observe that $[A]\left[P_{\mathcal{P}}^{0}\right][B]=[P]$, where $P_{\mathcal{P}}^{0}:=\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{array}\right)$. Since $\left[P_{\mathcal{P}}^{0}\right][B][\mathcal{Z}]=\left[\tilde{\mathcal{Z}}_{c_{1}, c_{2}}^{i, j, k}\right]$, then $[\mathcal{X}]=[A]\left[\tilde{\mathcal{Z}}_{c_{1}, c_{2}}^{i, j, k}\right]$. Observe that $[A] \in \mathcal{A}(2)$ and the direct statement is proved.
$(\Leftarrow)$ To prove the converse direction we assume that there exist $[A] \in \mathcal{A}(2)$, two real numbers $c_{1}$ and $c_{2}$, and a triplet of indices such that $(i, j, k) \in\{(1,2,3),(1,3,2),(2,3,1)\}$, such that $[\mathcal{X}]=[A]\left[\tilde{\mathcal{Z}}_{c_{1}, c_{2}}^{i, j, k}\right]$, where a planar curve $\tilde{\mathcal{Z}}_{c_{1}, c_{2}}^{i, j, k}$ is given by (7). Let $B$ be a matrix defined in the first part of the proof. A direct computation shows that $\mathcal{Z}$ is projected to $\mathcal{X}$ by the parallel projection $[P]=[A]\left[P_{\mathcal{P}}^{0}\right][B]$.

The families of curves $\tilde{\mathcal{Z}}_{c_{1}, c_{2}}^{i, j, k}$ given by (7) have a large overlap. The following corollary eliminates this redundancy and, therefore, is useful for practical computations. A proof can be found in [6].

Corollary 11. (reduced parallel projection criterion.) A curve $\mathcal{Z} \subset \mathbb{R}^{3}$ projects to $\mathcal{X} \subset \mathbb{R}^{2}$ by a parallel projection if and only if there exist $a_{1}, a_{2}, b \in \mathbb{R}$ such that the curve $\mathcal{X}$ is $\mathcal{A}(2)$-equivalent to one of the following planar curves:

$$
\begin{align*}
\tilde{\mathcal{Z}}_{a_{1}, a_{2}} & =\overline{\left\{\left(z_{1}+a_{1} z_{3}, z_{2}+a_{2} z_{3}\right) \mid\left(z_{1}, z_{2}, z_{3}\right) \in \mathcal{Z}\right\}}, \\
\tilde{\mathcal{Z}}_{b} & =\overline{\left\{\left(z_{1}+b z_{2}, z_{3}\right) \mid\left(z_{1}, z_{2}, z_{3}\right) \in \mathcal{Z}\right\}},  \tag{9}\\
\tilde{\mathcal{Z}} & =\overline{\left\{\left(z_{2}, z_{3}\right) \mid\left(z_{1}, z_{2}, z_{3}\right) \in \mathcal{Z}\right\}} .
\end{align*}
$$

## 4. GROUP-EQUIVALENCE PROBLEM

Theorems 9 and 10 reduce the projection problem to the problem of establishing group-action equivalence between a given curve and a curve from a certain family. A variety of methods exist to solve group-equivalence problem for curves. We base our algorithm on the differential signature construction described in [7]. Differential signature proposed there solves local equivalence problem for smooth curves. We adapt this construction to the algebraic setting and prove that differential signature gives a solution of global equivalence problem in the case of algebraic curves. For this purpose, we introduce a notion of a differentially separating set of rational invariants. We discuss the possibility of using some other methods for solving equivalence problem in Section 7.

### 4.1 Differential invariants for planar curves

A rational action of an algebraic group $G$ on $\mathbb{R}^{2}$ can be prolonged to an action on the $n$-th jet space $J^{n}=\mathbb{R}^{n+2}$ with coordinates $\left(x, y, y^{(1)}, \ldots, y^{(n)}\right)$ as follows. ${ }^{4}$ For a fixed $g \in G$, let $(\bar{x}, \bar{y})=g \cdot(x, y)$. Then $\bar{x}, \bar{y}$ are rational functions of $(x, y)$ and

$$
\begin{align*}
& g \cdot\left(x, y, y^{(1)}, \ldots, y^{(n)}\right)=\left(\bar{x}, \bar{y}, \bar{y}^{(1)}, \ldots, \bar{y}^{(n)}\right), \text { where } \\
& \bar{y}^{(1)}=\frac{\frac{d}{d x}[\bar{y}(x, y)]}{\frac{d}{d x}[\bar{x}(x, y)]} \text { and for } k=1, \ldots, n-1 \\
& \bar{y}^{(k+1)}=\frac{\frac{d}{d x}\left[\bar{y}^{(k)}\left(x, y, y^{(1)}, \ldots, y^{(k)}\right)\right]}{\frac{d}{d x}[\bar{x}(x, y)]} . \tag{10}
\end{align*}
$$

In (10), $\frac{d}{d x}$ is the total derivative, applied under assumption that $y$ is function of $x .^{5}$ We note that a natural projection $\pi_{k}^{n}: J^{n} \rightarrow J^{k}, k<n$ is equivariant with respect to action (10).
${ }^{4}$ Here $y=y^{(0)}$ and $J^{0}=\mathbb{R}^{2}$.
${ }^{5}$ We note the duality of our view of variables $y^{(k)}$. On one hand, they are viewed as independent coordinate functions on $J^{n}$. On the other hand, operator $\frac{d}{d x}$ is applied under

Definition 12. A function on $J^{n}$ is called a differential function. The order of a differential function is the maximum value of $k$ such that the function explicitly depends on the variable $y^{(k)}$.

A differential function which is invariant under action (10) is called a differential invariant.

Remark 13. Due to equivariant property of the projection $\pi_{k}^{n}: J^{n} \rightarrow J^{k}, k<n$, a differential invariant of order $k$ on $J^{k}$ can be viewed as a differential invariant on $J^{n}$ for all $n \geq k$.

Definition 14. Let $G$ act on $\mathbb{R}^{N}$. A set $\mathcal{I}$ of rational invariants is separating on a subset $W \subset \mathbb{R}^{N}$ if $W$ is contained in the domain of definition of each $I \in \mathcal{I}$ and $\forall w_{1}, w_{2} \in W$
$I\left(w_{1}\right)=I\left(w_{2}\right), \forall I \in \mathcal{I} \Longleftrightarrow \exists g \in G$ such that $w_{1}=g \cdot w_{2}$.
Definition 15. Let r-dimensional algebraic group $G$ act on $\mathbb{R}^{2}$. Let $K$ and $T$ be rational differential invariants of orders $r-1$ and $r$, respectively. The set $\mathcal{I}=\{K, T\}$ is called differentially separating if $K$ separates orbits on a Zariski open subset $W^{r-1} \subset J^{r-1}$ and $\mathcal{I}=\{K, T\}$ separate orbits on a Zariski open subset of $W^{r} \subset J^{r}$.

### 4.2 Jets of curves and signatures

In this section, we assume that $\mathcal{X} \subset \mathbb{R}^{2}$ is an irreducible algebraic curve, different from a vertical line. Let $F(x, y)$ be an irreducible polynomial, whose zero set equals to $\mathcal{X}$. Then the derivatives of $y$ with respect to $x$ are rational functions on $\mathcal{X}$, whose explicit formulas are obtained by implicit differentiation:

$$
\begin{aligned}
y_{\mathcal{X}}^{(1)} & =-\frac{F_{x}}{F_{y}}, \\
y_{\mathcal{X}}^{(2)} & =\frac{-F_{x x} F_{y}^{2}+2 F_{x y} F_{x} F_{y}-F_{y y} F_{x}^{2}}{F_{y}^{3}}, \ldots
\end{aligned}
$$

Definition 16. The $n$-th jet of a curve $\mathcal{X} \subset \mathbb{R}^{2}$ is a rational map $j_{\mathcal{X}}^{n}: \mathcal{X} \rightarrow J^{n}$, where for $\mathbf{p} \in \mathcal{X}$

$$
\begin{equation*}
j_{\mathcal{X}}^{n}(\mathbf{p})=\left(x(\mathbf{p}), y(\mathbf{p}), y_{\mathcal{X}}^{(1)}(\mathbf{p}), \ldots, y_{\mathcal{X}}^{(n)}(\mathbf{p})\right) \tag{11}
\end{equation*}
$$

From (10) it follows that $\quad j_{g \cdot \mathcal{X}}^{n}(g \cdot \mathbf{p})=g \cdot\left[j_{\mathcal{X}}^{n}(\mathbf{p})\right]$
Definition 17. A restriction of a rational differential function $\Phi: J^{n} \rightarrow \mathbb{R}$ to a curve $\mathcal{X}$ is a composition of $\Phi$ with the $n$-th jet of curve, i. e. $\left.\Phi\right|_{\mathcal{X}}=\Phi \circ j_{\mathcal{X}}^{n}$. If defined, such composition produces a rational function $\mathcal{X} \rightarrow \mathbb{R}$.

Definition 18. Let $\mathcal{I}=\{K, T\}$ be a differentially separating set of invariants for the G-action (see Definition 15). Then a point $\mathbf{p} \in \mathcal{X}$ is called $\mathcal{I}$-regular if: (1) $\mathbf{p}$ is a nonsingular point of $\mathcal{X}$; (2) $j_{\mathcal{X}}^{r-1}(\mathbf{p}) \in W^{r-1}$ and $j_{\mathcal{X}}^{r}(\mathbf{p}) \in W^{r}$; (3) $\left.\frac{\partial K}{\partial y^{(r-1)}}\right|_{j_{\mathcal{X}}^{r-1}(\mathbf{p})} \neq 0$ and $\left.\frac{\partial T}{\partial y^{(r)}}\right|_{j_{\mathcal{X}}^{r}(\mathbf{p})} \neq 0$.

An algebraic curve $\mathcal{X} \subset \mathbb{R}^{2}$ is called non-exceptional with respect to $\mathcal{I}$ if all but a finite number of its points are $\mathcal{I}$ regular.

Remark 19. If $\mathcal{X}$ is non-exceptional with respect to $\mathcal{I}$ then $\left.K\right|_{\mathcal{X}}$ and $\left.T\right|_{\mathcal{X}}$ are rational functions on $\mathcal{X}$. The $\mathcal{I}$ regular points of $\mathcal{X}$ are in the domain of the definition of these functions.
assumption that $y$ is a function of $x$ and, therefore, $y^{(k)}$ is also viewed as the $k$-th derivative of $y$ with respect to $x$. (The same duality of view appears in calculus of variations.)

Definition 20. Let $\mathcal{I}=\{K, T\}$ be a differentially separating set of invariants with respect to $G$-action and $\mathcal{X}$ be non-exceptional with respect to $\mathcal{I}$. The signature $\mathcal{S}_{\mathcal{X}}$ is the image of the rational map $\left.S\right|_{\mathcal{X}}: \mathcal{X} \rightarrow \mathbb{R}^{2}$ defined by $S_{\mathcal{X}}(\mathbf{p})=\left(\left.K\right|_{\mathcal{X}}(\mathbf{p}),\left.T\right|_{\mathcal{X}}(\mathbf{p})\right)$.

Theorem 21. (Group-equivalence criterion.) Assume that irreducible algebraic curves $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are nonexceptional with respect to differentially separating invariants $\mathcal{I}=(K, T)$ under $G$-action. Then $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are $G$ equivalent if and only if their signatures are equal:

$$
\mathcal{X}_{1} \cong_{G} \mathcal{X}_{2} \quad \Longleftrightarrow \quad \mathcal{S}_{\mathcal{X}_{1}}=\mathcal{S}_{\mathcal{X}_{2}}
$$

We make the following remarks before proving the theorem.
Remark 22. Let $\mathcal{I}=\{K, T\}$ be a set of differentially separating invariants (see Definition 15). Let $\mathcal{X} \subset \mathbb{R}^{2}$ be a non $\mathcal{I}$-exceptional curve defined by an irreducible implicit equation $F(x, y)=0$. Then $\left.K\right|_{\mathcal{X}}(x, y)=\frac{k_{1}(x, y)}{k_{2}(x, y)},\left.T\right|_{\mathcal{X}}=$ $\frac{t_{1}(x, y)}{t_{2}(x, y)}$, where $k_{1}, k_{2}$ and $t_{1}, t_{2}$ are pairs of polynomials with no non-constant common factors modulo $F$. Consider an ideal

$$
X:=\left\langle F, k_{2} \varkappa-k_{1}, t_{2} \tau-t_{1}, k_{2} t_{2} \sigma-1\right\rangle \subset \mathbb{R}[\varkappa, \tau, x, y, \sigma] .
$$

The algebraic closure $\overline{\mathcal{S}_{\mathcal{X}}}$ of the signature set $\mathcal{S}_{\mathcal{X}}$ is the variety of the radical of the elimination ideal $\hat{X}=X \cap \mathbb{R}[\varkappa, \tau]$.

Remark 23. We note that $\operatorname{dim} \overline{\mathcal{S}_{\mathcal{X}}}=0$ if and only if $K_{\mathcal{X}}$ and $T_{\mathcal{X}}$ are constant functions on $\mathcal{X}$ and $\operatorname{dim} \overline{\mathcal{S}_{\mathcal{X}}}=1$ otherwise. In the latter case, $\overline{\mathcal{S}_{\mathcal{X}}}$ is an algebraic planar curve with a single irreducible defining equation $\hat{S}_{\mathcal{X}}(\varkappa, \tau)=0$. The equality of signatures for two curves, $\mathcal{S}_{\mathcal{X}_{1}}=\mathcal{S}_{\mathcal{X}_{2}}$, implies $\hat{S}_{\mathcal{X}_{1}}(\varkappa, \tau)$ is equal up to a constant multiple to $\hat{S}_{\mathcal{X}_{2}}(\varkappa, \tau)$. The converse is true over $\mathbb{C}$, but not over $\mathbb{R}$, because the latter is not an algebraically closed field (see [10]).

Proof of Theorem 21. Direction $\Longrightarrow$ follows immediately from the definition of invariants. Below we prove $\Longleftarrow$. We notice that there are two cases. Either $\left.K\right|_{\mathcal{X}_{1}}$ and $\left.K\right|_{\mathcal{X}_{2}}$ are constant maps on $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$, respectively, and these maps take the same value. Otherwise both $\left.K\right|_{\mathcal{X}_{1}}$ and $\left.K\right|_{\mathcal{X}_{2}}$ are non-constant rational maps on $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$, respectively.

Case 1: There exists $c \in \mathbb{R}$ such that $\left.K\right|_{\mathcal{X}_{1}}\left(\mathbf{p}_{1}\right)=c$ and $K_{\mathcal{X}_{2}}\left(\mathbf{p}_{2}\right)=c$ for all $\mathbf{p}_{1} \in \mathcal{X}_{1}$ and for all $\mathbf{p}_{2} \in \mathcal{X}_{2}$. Since $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are non-exceptional, we may fix $\mathcal{I}_{G}$-regular points $\mathbf{p}_{1}=\left(x_{1}, y_{1}\right) \in \mathcal{X}_{1}$ and $\mathbf{p}_{2}=\left(x_{2}, y_{2}\right) \in \mathcal{X}_{2}$. Then, due to separation property of the invariant $K, \exists g \in G$ such that $j_{\mathcal{X}_{1}}^{r-1}\left(\mathbf{p}_{1}\right)=g \cdot\left[j_{\mathcal{X}_{2}}^{r-1}\left(\mathbf{p}_{2}\right)\right]$. We consider a new algebraic curve $\mathcal{X}_{3}=g \cdot \mathcal{X}_{2}$. Then due to (11), we have

$$
\begin{equation*}
j_{\mathcal{X}_{1}}^{r-1}\left(\mathbf{p}_{1}\right)=j_{\mathcal{X}_{3}}^{r-1}\left(\mathbf{p}_{1}\right)=: \mathbf{p}^{(r-1)} \tag{12}
\end{equation*}
$$

Since $\mathbf{p}_{1}$ is a $\mathcal{I}$-regular point of $\mathcal{X}_{1}$, it follows from (12) that it is also a $\mathcal{I}$-regular point of $\mathcal{X}_{3}$ and, in particular, is non-singular. Let $F_{1}(x, y)=0$ and $F_{3}(x, y)=0$ be implicit equations of $\mathcal{X}_{1}$ and $\mathcal{X}_{3}$, respectively. We may assume that $\frac{\partial F_{1}}{\partial y}\left(\mathbf{p}_{1}\right) \neq 0$ and $\frac{\partial F_{3}}{\partial y}\left(\mathbf{p}_{1}\right) \neq 0$ (otherwise, $\frac{\partial F_{1}}{\partial x}\left(\mathbf{p}_{1}\right) \neq 0$ and $\frac{\partial F_{3}}{\partial x}\left(\mathbf{p}_{1}\right) \neq 0$ and we may use a similar argument). Then, there exist functions $f_{1}(x)$ and $f_{3}(x)$, analytic on an interval $I \ni x_{1}$, such that $F_{1}\left(x, f_{1}(x)\right)=0$ and $F_{3}\left(x, f_{3}(x)\right)=0$ for $x \in I_{1}$.

Functions $y=f_{1}(x)$ and $y=f_{3}(x)$ are local analytic solutions of differential equation

$$
\begin{equation*}
K\left(x, y, y^{(1)}, \ldots, y^{(r-1)}\right)=c \tag{13}
\end{equation*}
$$

with the same initial condition $f_{1}^{(k)}\left(x_{1}\right)=f_{3}^{(k)}\left(x_{1}\right), k=$ $0, \ldots, r-1$ prescribed by (12). From the $\mathcal{I}$-regularity of $\mathbf{p}_{1}$, we have that $\left.\frac{\partial K}{\partial y^{(r-1)}}\right|_{\mathbf{p}^{(r-1)}} \neq 0$ and so (13) can be solved for $y^{(r-1)}$ :

$$
\begin{equation*}
y^{(r-1)}=H\left(x, y, y^{(1)}, \ldots, y^{(r-2)}\right) \tag{14}
\end{equation*}
$$

where function $H$ is smooth in a neighborhood $\mathbf{p}^{(r-1)} \in$ $J^{r-1}$. From the uniqueness theorem for the solutions of ODEs, it follows that $f_{1}(x)=f_{3}(x)$ on an interval $I \ni x_{1}$. Since $\mathcal{X}_{1}$ and $\mathcal{X}_{3}$ are irreducible algebraic curves it follows that $\mathcal{X}_{1}=\mathcal{X}_{3}$. Therefore, $\mathcal{X}_{1}=g \cdot \mathcal{X}_{2}$.

Case 2: $\left.K\right|_{\mathcal{X}_{1}}$ and $\left.K\right|_{\mathcal{X}_{2}}$ are non-constant rational maps. Then $\mathcal{S}_{\mathcal{X}_{1}}=\mathcal{S}_{\mathcal{X}_{2}}$ is a one-dimensional set that we will denote $\mathcal{S}$. Let $\hat{S}(\varkappa, \tau)=0$ be the implicit equation for $\mathcal{S}$ (see Remark 22). We know that $\frac{\partial \hat{S}}{\partial \tau}(\varkappa, \tau) \neq 0$ for all but finite number of values $(\varkappa, \tau)$, because, otherwise, $K \mid \mathcal{X}_{1}$ and $K \mid \mathcal{X}_{2}$ are constant maps. Therefore, since the curves are nonexceptional, there exists $\mathcal{I}$-regular points $\mathbf{p}_{1}=\left(x_{1}, y_{1}\right) \in \mathcal{X}_{1}$ and $\mathbf{p}_{2}=\left(x_{2}, y_{2}\right) \in \mathcal{X}_{2}$ such that

$$
\begin{gather*}
K\left|\mathcal{X}_{1}\left(\mathbf{p}_{1}\right)=K\right|_{\mathcal{X}_{2}}\left(\mathbf{p}_{2}\right)=: \varkappa_{0},\left.T\right|_{\mathcal{X}_{1}}\left(\mathbf{p}_{1}\right)=\left.T\right|_{\mathcal{X}_{2}}\left(\mathbf{p}_{2}\right)=: \tau_{0} \\
\text { and } \frac{\partial \hat{S}}{\partial \tau}\left(\varkappa_{0}, \tau_{0}\right) \neq 0 . \tag{15}
\end{gather*}
$$

Due to separation property of the set $\mathcal{I}_{G}=\{K, T\}, \exists g \in$ $G$ such that $j_{\mathcal{X}_{1}}^{r}\left(\mathbf{p}_{1}\right)=g \cdot\left[j_{\mathcal{X}_{2}}^{r}\left(\mathbf{p}_{2}\right)\right]$. We consider a new algebraic curve $\mathcal{X}_{3}=g \cdot \mathcal{X}_{2}$. Then due to (11), we have

$$
\begin{equation*}
j_{\mathcal{X}_{1}}^{r}\left(\mathbf{p}_{1}\right)=j_{\mathcal{X}_{3}}^{r}\left(\mathbf{p}_{1}\right)=: \mathbf{p}^{(r)} \tag{16}
\end{equation*}
$$

From (15), (16) and $\mathcal{I}$-regularity of the point $\mathbf{p}_{1} \in \mathcal{X}_{1}$ it follows that

$$
\begin{equation*}
K\left(\mathbf{p}^{(r)}\right)=\varkappa_{0}, \quad T\left(\mathbf{p}^{(r)}\right)=\tau_{0} \text { and }\left.\frac{\partial T}{\partial y^{(r)}}\right|_{\mathbf{p}^{(r)}} \neq 0 \tag{17}
\end{equation*}
$$

Since $\mathbf{p}_{1}$ is a $\mathcal{I}$-regular point of $\mathcal{X}_{1}$, it follows from (16) that it is also a $\mathcal{I}$-regular point of $\mathcal{X}_{3}$ and, in particular, is non-singular. Let $F_{1}(x, y)=0$ and $F_{3}(x, y)=0$ be implicit equations of $\mathcal{X}_{1}$ and $\mathcal{X}_{3}$, respectively. We may assume that $\frac{\partial F_{1}}{\partial y} \neq 0$ and $\frac{\partial F_{3}}{\partial y} \neq 0$ (otherwise, $\frac{\partial F_{1}}{\partial x} \neq 0$ and $\frac{\partial F_{3}}{\partial x} \neq 0$ and we may use a similar argument). Then, there exist functions $f_{1}(x)$ and $f_{3}(x)$, analytic on an interval $I \ni x_{1}$, such that $F_{1}\left(x, f_{1}(x)\right)=0$ and $F_{3}\left(x, f_{3}(x)\right)=0$ for $x \in I_{1}$.

Then functions $y=f_{1}(x)$ and $y=f_{3}(x)$ are local analytic solutions of differential equation

$$
\begin{equation*}
\hat{S}\left(K\left(x, y, y^{(1)}, \ldots, y^{(r-1)}\right), T\left(x, y, y^{(1)}, \ldots, y^{(r)}\right)\right)=0 \tag{18}
\end{equation*}
$$

with the same initial condition $f_{1}^{(k)}\left(x_{1}\right)=f_{3}^{(k)}\left(x_{1}\right), k=$ $0, \ldots, r$, dictated by (16).

Since $\frac{\partial \hat{S}}{\partial \tau}\left(\varkappa_{0}, \tau_{0}\right) \neq 0$ and $\left.\frac{\partial T}{\partial y^{(r)}}\right|_{\mathbf{p}^{(r)}} \neq 0$ (see (15) and (17)), equation (18) can be solved for $y^{(r)}$ :

$$
\begin{equation*}
y^{(r)}=H\left(x, y, y^{(1)}, \ldots, y^{(r-1)}\right) \tag{19}
\end{equation*}
$$

where function $H$ is smooth in a neighborhood $\mathbf{p}^{(r)} \in J^{r}$. From the uniqueness theorem for the solutions of ODE it follows that $f_{1}(x)=f_{3}(x)$ on an interval $I \ni x_{1}$. Since $\mathcal{X}_{1}$ and $\mathcal{X}_{3}$ are irreducible algebraic curves it follows that $\mathcal{X}_{1}=\mathcal{X}_{3}$. Therefore, $\mathcal{X}_{1}=g \cdot \mathcal{X}_{2}$.

### 4.3 Separating sets of invariants for affine and projective actions

In this section, we construct a differentially separating set of rational invariants for affine and projective actions. We will build them from classical invariants from differential geometry [8, 3]. We start with Euclidean curvature $\kappa=\frac{y^{(2)}}{\left(1+\left[y^{(1)}\right]^{2}\right)^{3 / 2}}$ which is, up to a $\operatorname{sign}^{6}$, a differential invariant of the lowest order. Higher order Euclidean differential invariants are obtained by differentiating the curvature with respect to the Euclidean arclength $d s=\sqrt{1+\left[y^{(1)}\right]^{2}} d x$,

Affine and projective curvatures and infinitesimal arclengths are well known, and can be expressed in terms of Euclidean invariants [11, 19]. In particular, $\mathcal{S} \mathcal{A}$-curvature $\mu$ and infinitesimal $\mathcal{S} \mathcal{A}$-arclength $d \alpha$ are expressed in terms of their Euclidean counterparts as follows:

$$
\begin{equation*}
\mu=\frac{3 \kappa\left(\kappa_{s s}+3 \kappa^{3}\right)-5 \kappa_{s}^{2}}{9 \kappa^{8 / 3}}, \quad d \alpha=\kappa^{1 / 3} d s \tag{20}
\end{equation*}
$$

By considering effects of scalings and reflections on $\mathcal{S} \mathcal{A}(2)$ invariants, we obtain two lowest order $\mathcal{A}(2)$-invariants:

$$
\begin{equation*}
K_{\mathcal{A}}=\frac{\left(\mu_{\alpha}\right)^{2}}{\mu^{3}}, \quad T_{\mathcal{A}}=\frac{\mu_{\alpha \alpha}}{3 \mu^{2}} \tag{21}
\end{equation*}
$$

They are of order 5 and 6 , respectively, and are rational functions in jet variables.
$\mathcal{P G} \mathcal{L}(3)$-curvature $\eta$ and infinitesimal arclength $d \rho$ are expressed in terms of their $\mathcal{S A}$-counterparts:

$$
\begin{equation*}
\eta=\frac{6 \mu_{\alpha \alpha \alpha} \mu_{\alpha}-7 \mu_{\alpha \alpha}^{2}-9 \mu_{\alpha}^{2} \mu}{6 \mu_{\alpha}^{8 / 3}}, \quad d \rho=\mu_{\alpha}^{1 / 3} d \alpha \tag{22}
\end{equation*}
$$

The two lowest order rational $\mathcal{P G \mathcal { L }}(3)$-invariants are of differential order 7 and 8 , respectively:

$$
\begin{equation*}
K_{\mathcal{P}}=\eta^{3}, \quad T_{\mathcal{P}}=\eta_{\rho} \tag{23}
\end{equation*}
$$

Theorem 24. According to Definition 15:
(1) The set $\mathcal{I}_{\mathcal{A}}=\left\{K_{\mathcal{A}}, T_{\mathcal{A}}\right\}$ given by (21) is differentially separating for the $\mathcal{A}(2)$-action on $\mathbb{R}^{2}$.
(2) The set $\mathcal{I}_{\mathcal{P G \mathcal { L }}}=\left\{K_{\mathcal{P}}, T_{\mathcal{P}}\right\}$ given by (23) is differentially separating for the $\mathcal{P G \mathcal { L }}(3)$-action on $\mathbb{R}^{2}$.

Proposition 25.
(1) $\mathcal{I}_{\mathcal{A}}$-exceptional algebraic curves are lines and parabolas.
(2) $\mathcal{I}_{\mathcal{P G \mathcal { L }}}$-exceptional algebraic curves are lines and conics.

Corollary 26. An $\mathcal{I}_{\mathcal{A}}$-exceptional algebraic curve is not $\mathcal{A}(2)$ equivalent to a non $\mathcal{I}_{\mathcal{A}}$-exceptional algebraic curve. An $\mathcal{I}_{\mathcal{P G L}}$-exceptional algebraic curve is not $\mathcal{P G \mathcal { L }}(3)$ equivalent to a non $\mathcal{I}_{\mathcal{P G \mathcal { L }}}$-exceptional algebraic curve.

Theorem 24, in combination with Theorem 21, leads to a solution for the projective and the affine equivalence problems for curves. For rather technical proofs of these theorems we refer a reader to [6]. See also [15] for related results for smooth curves.

[^4]
## 5. ALGORITHMS AND EXAMPLES

The algorithms for solving projection problems based on a combination of the projection criteria of Section 3 and the group equivalence criterion of Section 4.

### 5.1 Explicit formulas for invariants

Before stating the algorithms we write out explicit formulas for invariants in terms of jet coordinates. Let

$$
\begin{align*}
& \Delta_{1}=3 y^{(4)} y^{(2)}-5\left[y^{(3)}\right]^{2}  \tag{24}\\
& \Delta_{2}=9 y^{(5)}\left[y^{(2)}\right]^{2}-45 y^{(4)} y^{(3)} y^{(2)}+40\left[y^{(3)}\right]^{3}, \tag{25}
\end{align*}
$$

then a differentially separating set of rational $\mathcal{A}(2)$-invariants (21) is given by:

$$
\begin{align*}
K_{\mathcal{A}} & =\frac{\left(\Delta_{2}\right)^{2}}{\left(\Delta_{1}\right)^{3}}  \tag{26}\\
T_{\mathcal{A}} & =\frac{1}{\left(\Delta_{1}\right)^{2}}\left(9 y^{(6)}\left[y^{(2)}\right]^{3}-63 y^{(5)} y^{(3)}\left[y^{(2)}\right]^{2}\right. \\
& \left.-45\left[y^{(4)}\right]^{2}\left[y^{(2)}\right]^{2}+255 y^{(4)}\left[y^{(3)}\right]^{2} y^{(2)}-160\left[y^{(3)}\right]^{4}\right)
\end{align*}
$$

while a separating set of rational $\mathcal{P G \mathcal { L }}(3)$-invariants (23) is given by:

$$
\begin{aligned}
& K_{\mathcal{P}}=\frac{729}{8\left(\Delta_{2}\right)^{8}}\left(18 y^{(7)}\left[y^{(2)}\right]^{4} \Delta_{2}-189\left[y^{(6)}\right]^{2}\left[y^{(2)}\right]^{6}\right. \\
& +126 y^{(6)}\left[y^{(2)}\right]^{4}\left(9 y^{(5)} y^{(3)} y^{(2)}+15\left[y^{(4)}\right]^{2} y^{(2)}\right. \\
& \left.-25 y^{(4)}\left[y^{(3)}\right]^{2}\right)-189\left[y^{(5)}\right]^{2}\left[y^{(2)}\right]^{4}\left(4\left[y^{(3)}\right]^{2}\right. \\
& \left.+15 y^{(2)} y^{(4)}\right)+210 y^{(5)} y^{(3)}\left[y^{(2)}\right]^{2}\left(63\left[y^{(4)}\right]^{2}\left[y^{(2)}\right]^{2}\right. \\
& \left.-60 y^{(4)}\left[y^{(3)}\right]^{2} y^{(2)}+32\left[y^{(3)}\right]^{4}\right)-525 y^{(4)} y^{(2)} \\
& \left(9\left[y^{(4)}\right]^{3}\left[y^{(2)}\right]^{3}+15\left[y^{(4)}\right]^{2}\left[y^{(3)}\right]^{2}\left[y^{(2)}\right]^{2}\right. \\
& \left.\left.-60 y^{(4)}\left[y^{(3)}\right]^{4} y^{(2)}+64\left[y^{(3)}\right]^{6}\right)+11200\left[y^{(3)}\right]^{8}\right)^{3} \\
& T_{\mathcal{P}}=\frac{243\left[y^{(2)}\right]^{4}}{2\left(\Delta_{2}\right)^{4}}\left(2 y^{(8)} y^{(2)}\left(\Delta_{2}\right)^{2}-8 y^{(7)} \Delta_{2}\left(9 y^{(6)}\left[y^{(2)}\right]^{3}\right.\right. \\
& -36 y^{(5)} y^{(3)}\left[y^{(2)}\right]^{2}-45\left[y^{(4)}\right]^{2}\left[y^{(2)}\right]^{2}+120 y^{(4)}\left[y^{(3)}\right]^{2} \\
& \left.-40\left[y^{(3)}\right]^{4}\right)+504\left[y^{(6)}\right]^{3}\left[y^{(2)}\right]^{5}-504\left[y^{(6)}\right]^{2}\left[y^{(2)}\right]^{3} \\
& \left(9 y^{(5)} y^{(3)} y^{(2)}+15\left[y^{(4)}\right]^{2} y^{(2)}-25 y^{(4)}\left[y^{(3)}\right]^{2}\right) \\
& +28 y^{(6)}\left(432\left[y^{(5)}\right]^{2}\left[y^{(3)}\right]^{2}\left[y^{(2)}\right]^{3}+243\left[y^{(5)}\right]^{2} y^{(4)}\left[y^{(2)}\right]^{4}\right. \\
& -1800 y^{(5)} y^{(4)}\left[y^{(3)}\right]^{3}\left[y^{(2)}\right]^{2}-240 y^{(5)}\left[y^{(3)}\right]^{5} y^{(2)} \\
& +540 y^{(5)}\left[y^{(4)}\right]^{2}\left[y^{(3)}\right]\left[y^{(2)}\right]^{3}+6600\left[y^{(4)}\right]^{2}\left[y^{(3)}\right]^{4} y^{(2)} \\
& -2000 y^{(4)}\left[y^{(3)}\right]^{6}-5175\left[y^{(4)}\right]^{3}\left[y^{(3)}\right]^{2}\left[y^{(2)}\right]^{2} \\
& \left.+1350\left[y^{(4)}\right]^{4}\left[y^{(2)}\right]^{3}\right)-2835\left[y^{(5)}\right]^{4}\left[y^{(2)}\right]^{4} \\
& +252\left[y^{(5)}\right]^{3} y^{(3)}\left[y^{(2)}\right]^{2}\left(9 y^{(4)} y^{(2)}-136\left[y^{(3)}\right]^{2}\right) \\
& -35840\left[y^{(5)}\right]^{2}\left[y^{(3)}\right]^{6}-630\left[y^{(5)}\right]^{2}\left[y^{(4)}\right]\left[y^{(2)}\right] \\
& \left(69\left[y^{(4)}\right]^{2}\left[y^{(2)}\right]^{2}-160\left[y^{(3)}\right]^{4}-153 y^{(4)}\left[y^{(3)}\right]^{2}\left[y^{(2)}\right]\right) \\
& +2100 y^{(5)}\left[y^{(4)}\right]^{2} y^{(3)}\left(72\left[y^{(3)}\right]^{4}+63\left[y^{(4)}\right]^{2}\left[y^{(2)}\right]^{2}\right. \\
& \left.-193 y^{(4)}\left[y^{(3)}\right]^{2} y^{(2)}\right)-7875\left[y^{(4)}\right]^{4}\left(8\left[y^{(4)}\right]^{2}\left[y^{(2)}\right]^{2}\right. \\
& \left.\left.-22 y^{(4)}\left[y^{(3)}\right]^{2}\left[y^{(2)}\right]+9\left[y^{(3)}\right]^{4}\right)\right) .
\end{aligned}
$$

We adapt Definition 17 to rational curves as follows. Let $\mathcal{X}$ is a rational curve parametrized by $\gamma(t)=(x(t), y(t))$, such that $x(t)$ is not a constant function. ${ }^{7}$ Make a recursive

[^5]definition of the following rational functions of $t$ :
\[

$$
\begin{equation*}
y^{(1)}=\frac{\dot{y}}{\dot{x}} \quad, \ldots, \quad y^{(k)}=\frac{y^{(\dot{k-1)}}}{\dot{x}} \tag{28}
\end{equation*}
$$

\]

where denotes the derivative with respect to the parameter.
Let $\Phi$ be a rational differential function. Then the restriction of $\left.\Phi\right|_{\gamma}$ is computed by substituting (28) into $\Phi$. If defined, $\left.\Phi\right|_{\gamma}$ is a rational function of $t$. The following proposition follows from Proposition 25.

Proposition 27. Let $\mathcal{X}$ be a rational curve with a parameterization $\gamma(t)=(x(t), y(t))$, such that $x(t)$ is not a constant function. Then the restrictions $\left.\Delta_{1}\right|_{\gamma}$ and $\left.\Delta_{2}\right|_{\gamma}$ are rational functions of $t$.

If $\left.\Delta_{1}\right|_{\gamma}(t)$ is zero for more than a finite number of values $t$, then $\mathcal{X}$ is either a line or a parabola and $\left.\Delta_{1}\right|_{\gamma}(t)$ is zero for all $t$. Otherwise, restrictions $\left.K_{\mathcal{A}}\right|_{\gamma}$ and $\left.T_{\mathcal{A}}\right|_{\gamma}$ are rational functions of $t$.

If $\left.\Delta_{2}\right|_{\gamma}(t)$ is zero for more than a finite number of values $t$, then $\gamma$ is either a line or an irreducible conic and then $\left.\Delta_{2}\right|_{\gamma}(t)$ is zero for all $t$. Otherwise $\left.K_{\mathcal{P}}\right|_{\gamma}$ and $\left.T_{\mathcal{P}}\right|_{\gamma}$ are rational functions on $t$.

### 5.2 Central projections

The following algorithm is based on the central projection criterion stated in Theorem 9.

## Algorithm 28. (Central projections.)

INPUT: Parameterizations $\Gamma=\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{Q}(s)^{3}$ and $\gamma=$ $(x, y) \in \mathbb{Q}(t)^{2}$ of rational algebraic curves $\mathcal{Z} \subset \mathbb{R}^{3}$ and $\mathcal{X} \subset$ $\mathbb{R}^{2}$, respectively, such that $\dot{\Gamma} \times \ddot{\Gamma} \neq 0 .^{8}$
OUTPUT: The truth of the statement:

$$
\exists[P] \in \mathcal{C P}, \text { such that } \mathcal{X}=P(\mathcal{Z})
$$

## STEPS:

1. if $\left|\begin{array}{l}\dot{\gamma} \\ \ddot{\gamma}\end{array}\right| \underset{\mathbb{Q}(t)}{=0}$ then

$$
\text { if }\left|\begin{array}{l}
\dot{\Gamma} \\
\ddot{\dddot{ }} \\
\dddot{\Gamma}
\end{array}\right| \underset{\mathbb{Q}(s)}{ }=0
$$

then return TRUE
else return FALSE;
2. $\epsilon:=\left(\frac{z_{1}+c_{1}}{z_{3}+c_{3}}, \frac{z_{2}+c_{2}}{z_{3}+c_{3}}\right) \in \mathbb{Q}\left(c_{1}, c_{2}, c_{3}, s\right)^{2}$;
3. if $\left.\Delta_{2}\right|_{\gamma}=0$ then

$$
\begin{aligned}
& \text { if } \exists\left(c_{1}, c_{2}, c_{3}\right) \in \mathbb{R}^{3} \\
& z_{3}+c_{3} \underset{\mathbb{R}(s)}{\neq} 0 \wedge \left\lvert\, \begin{array}{l}
\dot{\epsilon} \\
\ddot{\epsilon}
\end{array} \underset{\mathbb{R}(s)}{\neq\left. 0 \wedge \Delta_{2}\right|_{\epsilon} \underset{\mathbb{R}(s)}{ } 0}\right.
\end{aligned}
$$

then return TRUE
else return FALSE.
4. return the truth of the statement:
$\exists\left(c_{1}, c_{2}, c_{3}\right) \in \mathbb{R}^{3}$

$$
z_{3}+\left.c_{3} \underset{\mathbb{R}(s)}{\neq 0} 0 \wedge\left|\begin{array}{c}
\dot{\epsilon}  \tag{29}\\
\ddot{\epsilon}
\end{array}\right| \underset{\mathbb{R}(s)}{\neq 0} 0 \wedge \Delta_{2}\right|_{\mathbb{R}(s)} ^{\neq 0} 0
$$

$\wedge \forall s \in \mathbb{R}$

$$
\left.\Delta_{2}\right|_{\epsilon} \neq 0 \Rightarrow \exists t \in \mathbb{R}
$$

$$
\left.\left.\left.\left.K_{\mathcal{P}}\right|_{\epsilon} \underset{\mathbb{R}}{ } K_{\mathcal{P}}\right|_{\gamma} \wedge T_{\mathcal{P}}\right|_{\epsilon} \underset{\mathbb{R}}{=} T_{\mathcal{P}}\right|_{\gamma} .
$$

[^6]Remark 29. In Algorithm 28, we use restrictions of differential functions to a family of curves parametrized by $\epsilon(c, s) \in \mathbb{Q}(c, s)^{2}$, where $c=\left(c_{1}, c_{2}, c_{3}\right)$ determines a member of the family and s serves to parametrize a curve in the family. In this case, derivatives in (28) are taken with respect to $s$.

Remark 30. On the first step of Algorithm 28, we consider the case when $\mathcal{X}$ is a line. Then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\mathcal{Z}$ is coplanar. If $\mathcal{X}$ is not a line we define, on Step 2, a rational map $\epsilon$ that parametrizes a family of curves. On Step 3, we consider the case when $\mathcal{X}$ is an irreducible conic. Then $\mathcal{Z}$ can be projected to $\mathcal{X}$ if and only if $\exists c$ such that the algebraic closure $\tilde{Z}_{c}$ of the image of $\epsilon(c, s)$ is an irreducible conic. If $\mathcal{X}$ is neither a line nor a conic we proceed to Step 4, where we decide if there exists $c \in \mathbb{R}^{3}$ such that (1) $\epsilon(c, s)$ is neither a line nor a conic ${ }^{9}$; (2) the signatures of the algebraic curve parametrized by $\gamma(t)$ and the curve $\tilde{Z}_{c}$ are the same.

Remark 31. If at least one of $\left.K_{\mathcal{P}}\right|_{\gamma}(t),\left.T_{\mathcal{P}}\right|_{\gamma}(t)$ is a nonconstant function, Step 4 of the algorithm can be performed as follows. Compute the implicit equation

$$
\begin{equation*}
\hat{S}_{\mathcal{X}}(\varkappa, \tau)=0 \tag{30}
\end{equation*}
$$

for the signature of $\mathcal{X}$ by eliminating $t$ from $\varkappa=\left.K_{\mathcal{P}}\right|_{\gamma}(t)$ and $\tau=\left.T_{\mathcal{P}}\right|_{\gamma}(t)$. Then substitute $\varkappa=\left.K_{\mathcal{P}}\right|_{\epsilon}(c, s)$ and $\tau=$ $\left.T_{\mathcal{P}}\right|_{\epsilon}(c, s)$ into (30). The left-hand side of (30) becomes a rational function, which we denote $H(c, s)$. Decide if there exists $c \in \mathbb{R}^{3}$ such that conditions (29) are satisfied, $\left.K_{\mathcal{P}}\right|_{\epsilon}(c, s)$, and $\left.T_{\mathcal{P}}\right|_{\epsilon}(c, s)$ are non-constant rational functions and the numerator of $H(c, s)$ is a zero polynomial in $s$. If we were solving the problem over $\mathbb{C}$, this would provide a sufficient condition for existence of a projection, but over $\mathbb{R}$ some additional steps have to be taken (see Remark 23). If both $\left.K_{\mathcal{P}}\right|_{\gamma}(t)$ and $\left.T_{\mathcal{P}}\right|_{\gamma}(t)$ are constant functions, then Step 4 can be preformed by deciding if there exists $c$, such that both $\left.K_{\mathcal{P}}\right|_{\epsilon}(c, s)$ and $\left.T_{\mathcal{P}}\right|_{\epsilon}(c, s)$ are constant functions with the same values as $\left.K_{\mathcal{P}}\right|_{\gamma}(t),\left.T_{\mathcal{P}}\right|_{\gamma}(t)$, respectively. An implementation over $\mathbb{C}$ is posted on the internet [25].

Remark 32. If the output is TRUE, then, in many cases, we can, in addition to establishing the existence of $c_{1}, c_{2}, c_{3}$ in Step 4 of the algorithm, find at least one of such triplets explicitly. We then know that $\mathcal{Z}$ can be projected to $\mathcal{X}$ by a projection centered at $\left(-c_{1},-c_{2},-c_{3}\right)$. We can also, in many cases, determine explicitly a transformation $[A] \in$ $\mathcal{P G \mathcal { L }}(3)$ that maps $\mathcal{X}$ to the algebraic closure $\tilde{Z}_{c}$ of the image of the map $\epsilon(c, s)$. We then know that $\mathcal{Z}$ can be projected to $\mathcal{X}$ by the projection $[P]=[A]\left[P_{\mathcal{C}}^{0}\right][B]$, where $P_{\mathcal{C}}^{0}$ and $B$ are defined by (6).

Example 33. We would like to decide if the spatial curve $\mathcal{Z}$ parametrized by

$$
\Gamma(s)=\left(s^{3}, s^{2}, s\right), s \in \mathbb{R}(\text { Twisted Cubic })
$$

projects to any of the four given planar curves $\mathcal{X}_{1}, \mathcal{X}_{2}, \mathcal{X}_{3}$

[^7]and $\mathcal{X}_{4}$ parametrized, respectively, by:
\[

$$
\begin{gathered}
\gamma_{1}(t)=\left(t^{2}, t\right), \quad \gamma_{2}(t)=\left(\frac{t^{3}}{t+1}, \frac{t^{2}}{t+1}\right) \\
\gamma_{3}(t)=\left(\frac{t}{t^{3}+1}, \frac{t^{2}}{t^{3}+1}\right), \quad \gamma_{4}(t)=\left(t, t^{5}\right) .
\end{gathered}
$$
\]

Let $\epsilon\left(c_{1}, c_{2}, c_{3}, s\right)=\left(\frac{s^{3}+c_{1}}{s+c_{3}}, \frac{s^{2}+c_{2}}{s+c_{3}}\right)$. Curve $\mathcal{X}_{1}$ is a parabola and so is $\mathcal{P G \mathcal { L }}(3)$-exceptional. It is known that all irreducible planar conics are $\mathcal{P G \mathcal { L }}(3)$-equivalent and so, from Theorem 9, we know that $\mathcal{Z}$ can be projected on $\mathcal{X}_{1}$ if there exist $c \in \mathbb{R}^{3}$, such that $\epsilon(c, s)$ parametrizes a conic. This is obviously true for $c_{1}=c_{2}=c_{3}=0$. Indeed, on can check that $\mathcal{Z}$ can be projected to $\mathcal{X}_{1}$ by projection $x=\frac{z_{1}}{z_{3}}, y=\frac{z_{2}}{z_{3}}$.

The curve $\mathcal{X}_{2}$ is not $\mathcal{P G \mathcal { L }}(3)$-exceptional. Its signature is parametrized by a constant map:

$$
\left.K_{\mathcal{P}}\right|_{\gamma_{2}}(t)=\frac{250047}{12800} \text { and }\left.T_{\mathcal{P}}\right|_{\gamma_{2}}(t)=0, \quad \forall t \in \mathbb{R}
$$

Following Algorithm 28, we need to decide whether there exists $c \in \mathbb{R}^{3}$, such that $\epsilon(c, s)$ does not parametrize a line or a conic and

$$
\left.K_{\mathcal{P}}\right|_{\epsilon}(s)=\frac{250047}{12800} \text { and }\left.T_{\mathcal{P}}\right|_{\epsilon}(s)=0, \forall s \in \mathbb{R}
$$

This is, indeed, true for $c_{1}=c_{2}=0$ and $c_{3}=1$. We can check that $\mathcal{Z}$ can be projected to $\mathcal{X}_{2}$ by the a central projection $x=\frac{z_{1}}{z_{3}+1}, y=\frac{z_{2}}{z_{3}+1}$.

The signature of $\mathcal{X}_{3}$ is parameterized by a non-constant map:

$$
\left.K_{\mathcal{P}}\right|_{\gamma_{3}}(t)=-\frac{9261}{50} \frac{t^{7}-t^{4}+t}{\left(t^{3}-1\right)^{8}},\left.T_{\mathcal{P}}\right|_{\gamma_{3}}(t)=-\frac{21}{10} \frac{\left(t^{3}+1\right)^{4}}{\left(t^{3}-1\right)^{4}}
$$

Evaluation of step 4 of Algorithm 28 yields TRUE. One can, in fact, check that for $c^{*}=(1,0,0)$, the map $\epsilon\left(c^{*}, s\right)$ does not parametrize a line or a conic and
$\left.K_{\mathcal{P}}\right|_{\epsilon}\left(c^{*}, s\right)=\left.K_{\mathcal{P}}\right|_{\gamma_{3}}(s)$ and $\left.T_{\mathcal{P}}\right|_{\epsilon}\left(c^{*}, s\right)=\left.K_{\mathcal{P}}\right|_{\gamma_{3}}(s), \forall s \in \mathbb{R}$.
We conclude that $\mathcal{Z}$ can be projected to $\mathcal{X}_{3}$. It is not difficult to determine a possible projection: $x=\frac{z_{3}}{z_{1}+1}, y=\frac{z_{2}}{z_{1}+1}$.

It is important to observe that although $\mathcal{Z}$ can be projected to each of the planar curves $\mathcal{X}_{1}, \mathcal{X}_{2}$, and $\mathcal{X}_{3}$, these planar curves are not $\mathcal{P G} \mathcal{L}(3)$-equivalent. This underscores an observation made in Remark 8.

The signature of $\mathcal{X}_{4}$ is parametrized by a constant map:

$$
\left.K_{\mathcal{P}}\right|_{\gamma_{4}}(t)=\frac{1029}{128} \text { and }\left.T_{\mathcal{P}}\right|_{\gamma_{4}}(t)=0, \quad \forall t
$$

Following Algorithm 28, we need to decide whether there exists $c \in \mathbb{R}^{3}$, such that $\epsilon(c, s)$ does not parametrize a line or a conic and

$$
\left.K_{\mathcal{P}}\right|_{\epsilon}(c, s)=\frac{1029}{128} \text { and }\left.T_{\mathcal{P}}\right|_{\epsilon}(c, s)=0, \forall s \in \mathbb{R}
$$

Substitution of several values of $s$ in the above equation yields a system of polynomial equations for $c_{1}, c_{2}, c_{3} \in \mathbb{R}$ that has no solutions. We conclude that there is no central projection from $\mathcal{Z}$ to $\mathcal{X}_{4}$.

### 5.3 Parallel projections

The algorithm for parallel projections is based on the reduced parallel projection criterion stated in Corollary 11. This algorithm follows the same logic but has more steps than Algorithm 28, because we need to decide whether a
given planar curve is $\mathcal{A}(2)$-equivalent to a curve parametrized by $\alpha(s)=\left(z_{2}(s), z_{3}(s)\right)$, or to a curve parametrized by $\beta(b, s)=\left(z_{1}(s)+b z_{2}(s), z_{3}(s)\right)$ for some $b \in \mathbb{R}$, or to a curve parametrized by $\delta\left(a_{1}, a_{2}, s\right)=\left(z_{1}(s)+a_{1} z_{3}(s), z_{2}+a_{2} z_{3}(s)\right)$ for some $a=\left(a_{1}, a_{2}\right) \in \mathbb{R}^{2}$. Since the affine transformations are considered, the differential function $\Delta_{2}$ is replaced with $\Delta_{1}$ (see (24)) and projective invariants are replaced with affine invariants (see (26)). Due to its similarity to Algorithm 28 , we refrain from writing out the steps of the parallel projection algorithm and content ourselves with presenting examples. The algorithm can be found in [6].

Example 34. To decide whether the spatial curve $\mathcal{Z}$ parametrized by

$$
\Gamma(s)=\left(s^{4}+1, s^{2}, s\right), s \in \mathbb{R}
$$

can be projected to $\mathcal{X}$ parametrized by

$$
\gamma(t)=\left(t, t^{4}+t^{2}\right), t \in \mathbb{R}
$$

by a parallel projection, we start by noticing that $\mathcal{X}$ is not an $\mathcal{A}(2)$-exceptional curve. Its signature is parametrized by a non-constant map:

$$
\left.T_{\mathcal{A}}\right|_{\gamma}(t)=\frac{100 t^{2}\left(3-14 t^{2}\right)^{2}}{\left(1-14 t^{2}\right)^{3}},\left.K_{\mathcal{A}}\right|_{\gamma}(t)=-5 \frac{\left(140 t^{4}-56 t^{2}+1\right)}{\left(1-14 t^{2}\right)^{2}} .
$$

We first check whether $\mathcal{X}$ is $\mathcal{A}(2)$-equivalent to a curve parametrized by $\alpha(s)=\left(z_{2}(s), z_{3}(s)\right)=\left(s^{2}, s\right)$. The answer is no, since $\alpha(s)$ defines a parabola, which is an $\mathcal{A}(2)$ exceptional curve and $\mathcal{X}$ is not $\mathcal{A}(2)$-exceptional. We next check if there exists $b \in \mathbb{R}$ such that $\mathcal{X}$ is $\mathcal{A}(2)$-equivalent to $a$ curve parametrized by $\beta_{b}(s)=\left(z_{1}(s)+b z_{2}(s), z_{3}(s)\right)=$ $\left(s^{4}+1+b s^{2}, s\right)$. We evaluate invariants (21):

$$
\begin{aligned}
\left.T_{\mathcal{A}}\right|_{\beta}(b, s) & =\frac{100 s^{2}\left(3 b-14 s^{2}\right)^{2}}{\left(b-14 s^{2}\right)^{3}} \\
\left.K_{\mathcal{A}}\right|_{\beta}(b, s) & =\frac{-5\left(140 s^{4}-56 b s^{2}+b^{2}\right)}{\left(b-14 s^{2}\right)^{2}}
\end{aligned}
$$

Since $\left.T_{\mathcal{A}}\right|_{\beta}(1, s)=\left.K_{\mathcal{A}}\right|_{\gamma}(s)$ and $\left.T_{\mathcal{A}}\right|_{\beta}(1, s)=\left.T_{\mathcal{A}}\right|_{\gamma}(s), \forall s$ we conclude that a curve parametrized by $\beta(1, s)$ is $\mathcal{A}(2)$ equivalent to $\mathcal{X}$ and, therefore, $\mathcal{Z}$ projects to $\mathcal{X}$ by a parallel projection.

Example 35. We would like to decide if the spatial curve $\mathcal{Z}$ parametrized by

$$
\Gamma(s)=\left(s^{3}, s^{2}, s\right), s \in \mathbb{R}(\text { Twisted Cubic) }
$$

projects to any of the three given planar curves $\mathcal{X}_{1}, \mathcal{X}_{2}$ and $\mathcal{X}_{3}$ parametrized, respectively, by:

$$
\begin{aligned}
\gamma_{1}(t) & =\left(t^{4}+t, t^{2}\right) \\
\gamma_{2}(t) & =\left(t^{3}-t, t^{3}+t^{2}\right) \\
\gamma_{3}(t) & =\left(\frac{t}{\left(1+t^{3}\right)}, \frac{t^{2}}{\left(1+t^{3}\right)}\right) \text { (Folium of Descartes) }
\end{aligned}
$$

None of the given planar curves is $\mathcal{A}(2)$-exceptional and so none of them is $\mathcal{A}(2)$-equivalent to a parabola parametrized by $\alpha(s)=\left(s^{2}, s\right)$. We then consider a family of curves parametrized by $\beta(b, s)=\left(s^{3}+b s^{2}, s\right)$ and establish that none of the curves in the family is $\mathcal{A}(2)$-equivalent to either of $\mathcal{X}$ 's. We proceed, by considering a family of curves $\delta\left(a_{1}, a_{2}, s\right)=\left(s^{3}+a_{1} s, s^{2}+a_{2} s\right)$ and establish that a curve parametrized by $\delta(0,1 / 2, s)$ is $\mathcal{A}(2)$-equivalent to $\mathcal{X}_{1}$ and a
curve parametrized by $\delta(0,0, s)$ is $\mathcal{A}(2)$-equivalent to $\mathcal{X}_{2}$, but there are no real values of $a_{1}$ and $a_{2}$, such that a curve parametrized by $\delta\left(a_{1}, a_{2}, s\right)$ is $\mathcal{A}(2)$-equivalent to $\mathcal{X}_{3}$.

We conclude that there are parallel projections of $\mathcal{Z}$ to both $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$, but not to $\mathcal{X}_{3}$. Note that $\mathcal{X}_{1}$ and $\mathcal{X}_{2}$ are not $\mathcal{A}(2)$-equivalent (their signatures have different implicit equations). This underscores an observation made in Remark 8.

## 6. PROJECTION PROBLEM FOR FINITE LISTS OF POINTS

In $[2,1]$, the authors present a solution to the problem of deciding whether or not there exists a parallel projection of a list $Z=\left(\mathbf{z}^{1}, \ldots, \mathbf{z}^{m}\right)$ of $m$ points in $\mathbb{R}^{3}$ to a list $X=$ $\left(\mathbf{x}^{1}, \ldots, \mathbf{x}^{m}\right)$ of $m$ points in $\mathbb{R}^{2}$, without finding a projection explicitly. They identify the lists $Z$ and $X$ with the elements of certain Grassmanian spaces and use Plüker embedding of Grassmanians into projective spaces to explicitly define the algebraic variety that characterizes pairs of sets related by a parallel projection.

We indicate here how our approach leads to an alternative solution for the projection problem for lists of points. Details of this adaptation appear in the dissertation [5]. The projection criterion of Theorem 9 adapt to finite lists of points as follows:

Theorem 36. (CEntral projection criterion for fiNITE LISTS.) A given list $Z=\left(\mathbf{z}^{1}, \ldots, \mathbf{z}^{m}\right)$ of $m$ points in $\mathbb{R}^{3}$ with coordinates $\mathbf{z}^{l}=\left(z_{1}^{l}, z_{2}^{l}, z_{3}^{l}\right), l=1 \ldots m$, projects onto a given list $X=\left(\mathbf{x}^{1}, \ldots, \mathbf{x}^{m}\right)$ of $m$ points in $\mathbb{R}^{2}$ with coordinates $\mathbf{x}^{l}=\left(x^{l}, y^{l}\right), l=1 \ldots m$, by a central projection if and only if there exist $c_{1}, c_{2}, c_{3} \in \mathbb{R}$ and $[A] \in \mathcal{P G \mathcal { L }}(3)$, such that

$$
\left[x^{l}, y^{l}, 1\right]^{\mathrm{T}}=[A]\left[z_{1}^{l}+c_{1}, z_{2}^{l}+c_{2}, z_{3}^{l}+c_{3}\right]^{\mathrm{T}} \text { for } l=1 \ldots m
$$

The proof of Theorem 36 is a straightforward adaptation of the proof of Theorem 9. The parallel projection criteria for curves, given in Theorem 10 and Corollary 11, are adapted to the finite lists in an analogous way.

The central and the parallel projection problems for lists of $m$ points is therefore reduced to a modification of the problems of equivalence of two lists of $m$ points in $\mathbb{P R}^{2}$ under the action of $\mathcal{P G \mathcal { L }}(3)$ and $\mathcal{A}(2)$ groups, respectively. A separating set of invariants for lists of $m$ points in $\mathbb{P R}^{2}$ under $\mathcal{A}(2)$-action consists of ratios of certain areas and is listed, for instance, in Theorem 3.5 of [20]. Similarly, a separating set of invariants for lists of $m$ ordered points in $\mathbb{P R}^{2}$ under $\mathcal{P G \mathcal { L }}(3)$-action consists of cross-ratios of certain areas and is listed, for instance, in Theorem 3.10 in [20]. In the case of central projections we, therefore, obtain a system of polynomial equations on $c_{1}, c_{2}$ and $c_{3}$ that have solutions if and only if the given set $Z$ projects to the given set $X$ and analog of Algorithms 28 follows. The parallel projections are treated in a similar way.

A solution of the projection problem for lists of points does not provide an immediate solution to the discretization of the projection problem for curves. Indeed, let $Z=$ $\left(\mathbf{z}^{1}, \ldots, \mathbf{z}^{m}\right)$ be a discrete sampling of a spatial curve $\mathcal{Z}$ and $X=\left(\mathbf{x}^{1}, \ldots, \mathbf{x}^{m}\right)$ be a discrete sampling of a planar curve $\mathcal{X}$. It might be impossible to project the list $Z$ onto $X$, even when the curve $\mathcal{Z}$ can be projected to the curve $\mathcal{X}$. Some approaches to the discretization of the projection algorithms for curves are discussed in the next section.

## 7. FURTHER RESEARCH

The projection criteria developed in Section 3 reduce the problem of object-image correspondence for curves under a projection from $\mathbb{R}^{3}$ to $\mathbb{R}^{2}$ to a variation of the groupequivalence problem for curves in $\mathbb{R}^{2}$. We use differential invariants to address the group-equivalence problem. In practical applications, curves are often given by samples of points. In this case, invariant numerical approximations of differential invariants presented in [7, 4] may be used. Differential invariants and their approximations are highly sensitive to image perturbations and, therefore, are not practical in many situations. Other types of invariants, such as semi-differential (or joint) invariants [23, 20] and integral invariants $[21,16,13]$ are less sensitive to image perturbations and may be employed to solve the group-equivalence problem.

One of the essential contributions of $[2,1]$ is the definition of an object/image distance between ordered sets of $m$ points in $\mathbb{R}^{3}$ and $\mathbb{R}^{2}$, such that the distance is zero if and only if these sets are related by a projection. Since, in practice, we are given only an approximate position of points, a "good" object/image distance provides a tool for deciding whether a given set of points in $\mathbb{R}^{2}$ is a good approximation of a projection of a given set of points in $\mathbb{R}^{3}$. Defining such object/image distance in the case of curves is an important direction of further research.

Our algorithm establishes existence of a projection, but does not compute a possible projection map (see Remark 32). Algorithmic reconstruction of a projection is another interesting question to consider.

## 8. ACKNOWLEDGEMENTS

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[24] http://en.wikipedia.org/wiki/File:Pinholecamera.svg
[25] http://www.math.ncsu.edu/~iakogan/symbolic/ projections.html


[^0]:    *Supported in part by NSF grants CCF-0728801 and CCF0347506.
    ${ }^{\dagger}$ Supported in part by NSF grant CCF-0728801 and NSA grant H98230-11-1-0129.

[^1]:    ${ }^{1}$ It is clear from (1) that multiplication of $P$ by a non-zero constant does not change the projection map. Therefore, we can identify $P$ with a point of the projective space $\mathbb{P R}^{11}$, rather than a point in $\mathbb{R}^{12}$. However, since we do not know which of the parameters are non-zero, in computations we have to keep all 12 parameters.

[^2]:    ${ }^{2}$ We include parameters $t$ and $s$ into the count.

[^3]:    ${ }^{3}$ Parallel projections are also called generalized weak perspective projections $[2,1]$.

[^4]:    ${ }^{6}$ The sign of $\kappa$ changes when a curve is reflected, rotated by $\pi$ radians or traced in the opposite direction. A rational function $\kappa^{2}$ is invariant under the full Euclidean group.

[^5]:    ${ }^{7}$ Equivalently, $\mathcal{X}$ is not a vertical line.

[^6]:    $\overline{{ }^{8} \text { Equivalently, } \mathcal{Z} \text { is not a line. }}$

[^7]:    ${ }^{9}$ For such values of $c$, we can prove that whether we first specialize a value of $c$ in $\epsilon(c, s)$ and then compute rational functions $\left.K_{\mathcal{P}}\right|_{\epsilon}(c, s)$ and $\left.T_{\mathcal{P}}\right|_{\epsilon}(c, s)$ or we perform these operations in the reverse order, we obtain the same rational functions of $s$.

