Inductive Construction of Moving Frames

Irina A. Kogan

Abstract. This paper presents a useful variation on the moving frame construction, which allows us to use a moving frame for a subgroup A of a Lie group G to produce a moving frame for the entire group G. This algorithm is applicable when G factors as a product of two subgroups $G = A \cdot B$ and automatically produces functional relations among invariants of G and its factors.

1. Introduction

Elie Cartan's method of equivalence [**4**] is a natural development of the Felix Klein Erlangen program (1872), which describes geometry as the study of invariants of group actions on geometric objects. Classically, a moving frame is an equivariant map from the space of submanifolds (or more rigorously, from the corresponding jet bundle) to the bundle of frames. Exterior differentiation of this map produces a number of differential invariants. Differential invariants provide a key to the solution of many equivalence problems and are also used in the process of reduction of differential equations and variational problems (see for instance [**5**], [**10**], [**15**], [**17**], [**2**] and [**16**]).

Considering moving frame constructions on homogeneous spaces, Griffiths [**13**] and Green [**12**] observed that a moving frame can be viewed as an equivariant map from the space of submanifolds to the group itself. Adopting this observation as a general definition of a moving frame, Fels and Olver [**8**], [**9**] generalized the Cartan's method to arbitrary, not necessarily transitive, finite-dimensional Lie group actions on a manifold, introducing a simple algorithm for constructing moving frames and differential invariants. According to this algorithm the moving frame construction reduces to solving a system of algebraic equations. This last step might become trivial or very difficult depending on the group action we consider.

Not surprisingly, the construction of moving frames and differential invariants is simpler when the acting group has fewer parameters. Thus, it is desirable to use the results obtained for a subgroup $A \subset G$ to construct a moving frame and differential invariants for the entire group G. The inductive algorithm presented here allows us, in the case when the group G factors as a product, to extend a moving frame for a subgroup to the entire group. As a byproduct one obtains at the same time the relations among the invariants of group G and its subgroup A. It worth remarking

Supported in part by NSF Grant DMS 99–83403.

that, in order to obtain such relations, the algorithm does not require the explicit fromulae for the invariants of either G or A, but only the corresponding moving frames (or normalizations) which lead to these invariants.

We illustrate the algorithm by making an induction from the Euclidean action on plane curves to the special affine action, and then to the action of the entire projective group, which leads to the expression of the affine invariants in terms of the Euclidean ones and the projective invariants in terms of the affine ones. These are classical actions whose differential invariants are well known (for instance, see [**3**], [**6**], [**8**]). The actions of all three groups play an important role in computer image processing [**7**], [**19**]. An alternative derivation of the affine curvature in terms of the Euclidean appears in [**3**], while the expression for the projective curvature in terms of the affine can be found in [**7**]. Some higher dimensional examples, including the derivation of the affine invariants in terms of the Euclidean for curves in $\mathbb{R}^3$, were also computed, but not included here to keep the presentation short.

2. The Method of Moving Frames

Given a manifold M of dimension m and an integer $1 \le p \le m$, we let $J^k = J^k(M,p)$ denote the k-th order jet bundle, whose fiber over $z \in M$ consists of equivalence classes of p-dimensional submanifolds of M under the equivalence relation of k-th order contact at z. The infinite jet bundle $J^\infty = J^\infty(M,p)$ is defined as the inverse limit of the finite jet bundles under the standard projections $\pi_k^{k+1}\colon J^{k+1} \to J^k$. We will identify functions and differential forms on J^k with their pull backs to any higher order jets including J^∞.

Let U be a coordinate chart on M. We arbitrarily divide the set of coordinate functions on U into two subsets: the set of independent variables $x^1, ..., x^p$ and the set of dependent variables $u^1, \dots u^q$, where $p + q = m$. The k-th jets of all submanifolds $S \subset U$ which satisfy the transversality condition $dx^1 \wedge \cdots \wedge dx^p|_S \neq 0$ form a coordinate chart $U^k \subset J^k$ which can be parameterized by coordinate functions $x^1, \dots, x^p$, u^α_J, where $i = 1, \dots, p$, $\alpha = 1, \dots, q$ and $J = (j_1, \dots, j_k)$, with $0 \le j_\nu \le p$, is a symmetric multi-index of length $|J| = k$.

The cotangent bundle over J^∞ has a distinguished sub-bundle $\mathcal{C}$, whose sections are identically zero when restricted to a jet of any p-dimensional submanifold of M. In local coordinates $\mathcal{C}$ is spanned by the forms $\theta^\alpha_J = du^\alpha_J - \sum_i^p u^\alpha_{J,i} dx^i$, $\alpha = 1, \dots, q$, $0 \le |J|$. The differential ideal generated by one-forms in $\mathcal{C}$ is called *contact ideal*. On a local chart we can define a complementary horizontal sub-bundle H spanned by the forms $dx^1, \dots, dx^p$. This splitting induces a bigrading on the algebra of differential forms $\bigwedge T^* J^\infty$. For any differential form λ, we let $\pi_H \lambda$ denote its purely horizontal component and $\pi_V \lambda$ denote its purely contact component. There is also a corresponding splitting of the tangent bundle over J^∞. In particular, the vector fields on J^∞, which are annihilated by any contact form, form a sub-bundle of *total* (or horizontal) vector fields.

A smooth action of a Lie group G on M can be uniquely prolonged to a smooth action on J^∞ under the condition that it preserves contact ideal. By definition, a k-th order *differential invariant* of G is a function on J^k which is invariant under the prolonged action.

We will review the basic steps of the moving frame construction presented in [**9**].

Definition 2.1. A k-th order (right) moving frame is a smooth right G-equivariant map $\rho^{(k)}$ from an open subset of J^k to G:

$$\rho^{(k)}(g \cdot z^{(k)}) = \rho^{(k)}(z^{(k)}) \cdot g^{-1}, \tag{1}$$

for all $g \in G$ such that $z^{(k)}$ and $g \cdot z^{(k)} \in J^k$ are in the domain of definition of ρ.

The existence of a moving frame on a jet bundle can be deduced from the following two theorems. See [**9**] for the proof of the first theorem and [**18**], [**16**] for the proof of the second one.

Theorem 2.2. *Let a Lie group G act on a manifold N. Then there exists a G-equivariant map from a neighborhood of each point in N to the group G if and only if G acts freely and regularly.*

The last condition of the theorem means that every point of N has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

Theorem 2.3. *Let G be a Lie group that acts locally effectively on each open subset of M. Then there is a minimal order $n \leq r = \dim G$, such that the prolonged action of G on J^k is locally free on some open and dense subset $\mathcal{V}^k \subset J^k$ for each $k \geq n$.*

By definition, *local freeness* of the action means that the isotropy group of each point is discrete. The order n in the theorem above is called the *order of stabilization* and the subsets $\mathcal{V}^k$ are called *regular.*

We notice that the conclusion of the second theorem is weaker than the assumption of the first one. It guarantees, however, that all orbits on $\mathcal{V}^n$ have the same dimension $r = \dim G$. Using the Frobenius theorem, one can construct a submanifold $\mathcal{K}^n$, which is transversal the orbits on an open neighborhood of a point $z^{(n)} \in \mathcal{V}^n$, and has complementary dimension. Such manifold is called a *cross-section* to the orbits. (A similar construction appears in the proof of Theorem 3.4 below.) If the action is regular then by shrinking $\mathcal{K}^n$ we can make it intersect each orbit no more than once. Let us assume for a moment that the action is free and regular. Then the moving frames (1) near $\mathcal{K}^n$ is defined by the condition

$$\rho(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}^n$$

Since each orbit intersects $\mathcal{K}$ at a unique point, then

$$\rho(z^{(n)}) \cdot z^{(n)} = \rho(g \cdot z^{(n)}) \cdot (g \cdot z^{(n)}) \tag{2}$$

and this leads to the right equivariance condition (1) due to the freeness assumption.

Remark 2.4. For a locally free, not necessarily regular action on $\mathcal{V}^n$, which is guaranteed by Theorem (2.3), a moving frame can be defined in a similar fashion. In this case, however, the equivariant condition (1) will hold only when g belongs to some open neighborhood of the identity in G which may depend on $z^{(n)}$.

The cross-section $\mathcal{K}^n$ and the moving frame ρ can be extended to any higher order regular set $\mathcal{V}^k$, including $\mathcal{V}^\infty \subset J^\infty$, by defining $\mathcal{K}^k = \{z^{(k)} | \pi^k_n z^{(k)} \in \mathcal{K}^n\}$ and $\rho(z^{(k)}) = \rho\left(\pi^k_n(z^{(k)})\right)$ for $k = n, \ldots, \infty$.

Remark 2.5. *Despite the locality of the moving frame definition, we will adopt a global notation, therefore writing $\rho\colon J^\infty \to G$, while, in fact, the domain and the range of ρ are some open subsets of J^∞ and G respectively.*

Given a moving frame, one can define a process of invariantization (see [**9**], [**16**]) which will project the space of differential forms (in particular functions) on J^∞ onto the space of invariant differential forms (functions). We start by lifting the prolonged G action to the space $\mathcal{B} = G \times J^\infty$:

$$h \cdot (g, z^{(\infty)}) = (gh^{-1}, h \cdot z^{(\infty)}),$$

where g, $h \in G$. We also introduce maps $w\colon \mathcal{B} \to J^\infty$ to be defined by the prolonged group action: $w(g, z^{(\infty)}) = g \cdot z^{(\infty)}$, and $\sigma\colon J^\infty \to \mathcal{B}$ to be defined via a moving frame: $\sigma(z^{(\infty)}) = (\rho(z^{(\infty)}), z^{(\infty)})$. We note that w is a G-invariant map, while σ is a G-equivariant map. Thus their composition $w \circ \sigma(z^{(\infty)}) = \rho(z^{(\infty)}) \cdot z^{(\infty)}$ is a G-invariant projection $J^\infty \longrightarrow \mathcal{K}^\infty$.

The cotangent bundle $T^*\mathcal{B}$ over $\mathcal{B}$ is a direct sum of the bundles T^*G and T^*J^∞. This induces a bigrading on $\bigwedge T^*\mathcal{B}$. For a differential form $\tilde\lambda$ on $\mathcal{B}$ we let $\pi_G\tilde\lambda$ denote the purely group component of $\tilde\lambda$ and $\pi_J\tilde\lambda$ its purely jet component. If $\tilde\lambda$ is a one-form then $\tilde\lambda = \pi_G\tilde\lambda + \pi_J\tilde\lambda = \pi_G\tilde\lambda + \pi_H\tilde\lambda + \pi_V\tilde\lambda$.

Definition 2.6. The invariantization of a differential form λ on J^∞ is the invariant differential form

$$\iota(\lambda) = \sigma^*\left(\pi_J(w^*\lambda)\right). \tag{3}$$

In the case of functions (zero forms) (3) reduces to

$$\iota(f)(z^{(\infty)}) = \sigma^* w^*(f)(z^{(\infty)}) = f\left(\rho(z^{(\infty)}) \cdot z^{(\infty)}\right). \tag{4}$$

Geometrically, invariantization of a differential form λ (or function f) is the unique invariant differential form (function) which agrees with λ (or f) on the cross-section $\mathcal{K}^\infty$. We note also that both $w^*\lambda$ and $\pi_J w^*\lambda$ are invariant forms on $\mathcal{B}$.

Invariantization of the coordinate functions:

$$H^i = \iota(x^i),\ i = 1, \ldots, p, \qquad I^\alpha_J = \iota(u^\alpha_J),\ \alpha = 1, \ldots, q,$$

provide a complete (or fundamental) set of local differential invariants on J^∞, in a sense that every other local differential invariant can be expressed as a function of these invariants.

Invariantization of the basis one-forms $dx^1, ..., dx^p$, θ^α_J:

$$\begin{aligned} \varpi^i &= \iota(dx^i) = \sigma^*\, d_J\, w^*\,(x^i),\ i = 1, \ldots, p, \\ \vartheta^\alpha_J &= \iota(\theta^\alpha_J) = \sigma^*\, \pi_J\, w^*\,(\theta^\alpha_J),\ \alpha = 1, \ldots, q \end{aligned}$$

produces an invariant coframe on J^∞. We note that invariantization preserves contact the sub-bundle $\mathcal{C}$ of T^*J^∞, but the horizontal sub-bundle H is not generally preserved under invariantization. We can decompose $\varpi^i = \iota(dx^i) = \omega^i + \eta^i$, $i = 1, \ldots, p$, where the non-zero horizontal forms

$$\omega^i = \sigma^*\, \pi_H\, w^*(dx^i) = \sigma^*\, d_H\, w^*\,(x^i) \tag{5}$$

are invariant up to a contact form, that is, $g^*\omega^i = \omega^i + \Theta^i$, for some contact one-forms Θ^i. Forms with such transformation property are called *contact invariant.* By adding contact forms η^i to ω^i one obtains fully invariant forms ϖ^i. Forms $\omega^i, i = 1, \ldots, p$ are linearly independent. The total vector fields $\mathcal{D}_i$, $i = 1, \ldots, p$, dual to ω^i, form a complete set of *invariant differential operators*, which map differential invariants to differential invariants of higher order. See [**17**] for further details.

Example 2.7. Let us consider the action of the special Euclidean group $SE(2) = SO(2) \ltimes R^2$ on plane curves $u = u(x)$. Its first prolongation, given by

$$
\begin{aligned}
x &\mapsto \cos(\alpha)x - \sin(\alpha)u + a, \\
u &\mapsto \sin(\alpha)x + \cos(\alpha)u + b, \\
u_x &\mapsto \frac{\sin(\alpha) + \cos(\alpha)u_x}{\cos(\alpha) - \sin(\alpha)u_x}
\end{aligned} \tag{6}
$$

defines a free action on $J^1(\mathbb{R}^2, 1)$. A moving frame on $J^1(\mathbb{R}^2, 1)$ can be obtained by choosing a cross-section $\{x = 0,\ u = 0,\ u_x = 0\}$. Then an equivariant map $J^1(\mathbb{R}^2, 1) \to SE(2)$ is found by setting expressions (6) equal to zero and solving for the groups parameters:

$$
\alpha = -\arctan(u_x), \qquad a = -\frac{u_x u + x}{\sqrt{1+u_x^2}}, \qquad b = \frac{u_x x - u}{\sqrt{1+u_x^2}}. \tag{7}
$$

The corresponding element of the special Euclidean group can be written in a matrix form:

$$
\rho = \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.
$$

A fundamental set of k-th order differential invariants can be obtained by prolonging the action to J^k and normalizing the group parameters, that is by substituting (7) into the formulae. For instance, the forth order prolongation is given by:

$$
\begin{aligned}
u_{xx} &\mapsto \frac{u_{xx}}{\Delta^3}, \\
u_{xxx} &\mapsto \frac{\Delta u_{xxx} + 3\sin(\alpha)u_{xx}^2}{\Delta^5}, \\
u_{xxxx} &\mapsto \frac{\Delta^2 u_{xxxx} + 10\sin(\alpha)\Delta u_{xx}u_{xxx} + 15\sin^2(\alpha)u_{xx}^3}{\Delta^7},
\end{aligned} \tag{8}
$$

where $\Delta = \cos(\alpha) - \sin(\alpha)u_x$. Substitution of (7) into (8) produces fourth order differential invariants:

$$
\begin{aligned}
I_2^e &= \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \\
I_3^e &= \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \\
I_4^e &= \frac{(1+u_x^2)^2 u_{xxxx} - 10u_x u_{xx} u_{xxx}(1+u_x^2) + 15u_x^2 u_{xx}^3}{(1+u_x^2)^{9/2}}.
\end{aligned} \tag{9}
$$

We note that $I_2^e = \kappa$, the Euclidean curvature, $I_3^e = \kappa_s = \frac{d\kappa}{ds}$, where $ds = \sqrt{1+u_x^2}\,dx$ is infinitesimal arc length, but $I_4^e = \kappa_{ss} + 3\kappa^3$ (instead of just κ_{ss}), according to recurrence formula (13.4) in [**9**]. The contact invariant differential form equals to $\omega = \sigma^*(d_H\, w^* x) = \sqrt{1+u_x^2}\,dx = ds$. The dual total vector field $\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}} D_x = \frac{d}{ds}$ provide an invariant differential operator, such that any other invariant can be expressed as a function of κ and its derivatives with respect to $\mathcal{D}$.

Remark 2.8. Lack of space precludes us from a detailed comparison of the classical method of moving frames as presented, for instance, in [**4**] and [**14**], with its generalization [**9**] described above. We note, however, that all classical moving frames lead to equivariant maps from a jet bundle to the group under consideration, while certainly not every such map can be described as an invariant section of a frame bundle. Classical differential invariants are obtained by pulling back the invariant coframe on G under this equivariant map, which may lead to a different (but equivalent) set of fundamental invariants.

3. A Moving Frame Construction for a Group that Factors as a Product.

We say that a group G factors as a product of its subgroups A and B if $G = A{\cdot}B$, that is, for any $g \in G$ there are $a \in A$ and $b \in B$ such that $g = ab$. We reproduce two useful statements from [**11**].

Theorem 3.1. *Let G be a group, and let A and B be two subgroups of G. Then the following conditions are equivalent:*
a) the reduction of the natural action of G on G/B to A is transitive,
b) $G = A \cdot B$,
c) $G = B \cdot A$,
d) the reduction of the natural action of G on G/A to B is transitive.

Corollary 3.2. *The reduction of the natural action of G on G/B to A is free and transitive if and only if $G = A \cdot B$ (or $G = B \cdot A$) and $A \cap B = e$.*

Remark 3.3. If $G = A \cdot B$ and $A \cap B = e$, then for each $g \in G$ there are *unique* elements $a \in A$ and $b \in B$ such that $g = ab$. In this case the manifolds $A \times B$ and G are diffeomorphic (although they are *not* in general isomorphic as groups). In the case when $A \cap B$ is discrete then $A \times B$ is locally diffeomorphic to G.

The following theorem plays a central role in our construction.

Theorem 3.4. *Let A and B act regularly on a manifold M, and assume that in a neighborhood U of a point $z_0 \in M$ the infinitesimal generators of the A-action are linearly independent from the generators of the B-action. Then locally there exists a submanifold $\mathcal{K}_A$ through the point z_0, which is transverse to the orbits of the subgroup A and is invariant under the action of the subgroup B.*

Proof. Let a be the dimension of the A-orbits, b be the dimension of the B-orbits on U and $m = \dim M$. By Frobenius' theorem we can locally rectify the orbits of B, that is, we can introduce coordinates $\{y_1, \dots, y_b, x_1, \dots, x_{m-b}\}$ such that the orbits of B are defined by the equations $x_i = k_i$, $i = 1, \dots, m-b$, where k_i are some constants. The orbits of B are integral manifolds for the distribution $\{\frac{\partial}{\partial y_1} \dots \frac{\partial}{\partial y_b}\}$. The functions x_i are invariant under the B-action. Let vector fields $X_1, \dots, X_a$ and $Y_1, \dots, Y_b$ be a basis for infinitesimal generators of the action of A and B respectively in a neighborhood U containing z_0 . The vector fields $Y_i, i = 1, \dots, b$ and $\frac{\partial}{\partial x_j}$, $j = 1 \dots m-b$ are linearly independent by the choice of coordinates, and their union forms a basis in TU. We can choose $c = m-b-a$ vector fields $\frac{\partial}{\partial x_{j_1}} \dots \frac{\partial}{\partial x_{j_c}}$ which are linearly independent from $X_1, \dots, X_a$ in TU. Let $\mathcal{K}$ be an integral manifold through the point z_0 for the involutive distribution $\Delta = \{\frac{\partial}{\partial x_{j_1}} \dots \frac{\partial}{\partial x_{j_c}}, \frac{\partial}{\partial y_1} \dots \frac{\partial}{\partial y_b}\}$. By construction $\mathcal{K}_A$ is a union of orbits of B and thus is invariant under the action of B. On the other hand, the distribution Δ is transverse to the infinitesimal generators $X_1, \dots, X_a$ of the A-action, and so is transverse to the orbits of A. $\square$

With this result we construct a moving frame for a product of groups A and B as follows.

ALGORITHM 3.5. Let $G = A\cdot B$ and let $A\cap B$ be discrete. Then, as a manifold, G is locally diffeomorphic to $A \times B$. Let n be the order of stabilization of the G-action. Since both A and B act locally freely on $\mathcal{V}^n \subset J^n$ and their intersection is discrete then the infinitesimal generators of the A-action and the B-action are linearly independent at each point of $\mathcal{V}^n$ and hence they satisfy the conditions of Theorem 3.4. Thus there is a cross-section $\mathcal{K}^n_A \subset \mathcal{V}^n$ for the action of A which is invariant under the action of B. We use this cross-section to construct a moving frame ρ_A for A. The map $\rho_A(z^{(n)})\cdot z^{(n)}$ projects $\mathcal{V}^n$ on the cross-section $\mathcal{K}^n_A$, which is invariant under the action of B. Moreover the action of B on $\mathcal{K}^n_A$ is locally free and hence we can choose a cross-section $\mathcal{K}^n \subset \mathcal{K}^n_A$ that defines a moving frame $\rho_B : \mathcal{K}^n_A \to B$. We can extend ρ_B to a map $\tilde{\rho}_B : \mathcal{V}^n \to B$ by the formula

$$\tilde{\rho}_B(z^{(n)}) = \rho_B\left(\rho_A(z^{(n)})\cdot z^{(n)}\right). \tag{10}$$

The map $\tilde{\rho}_B$ is A-invariant but, in contrast to ρ_B, it is not B-equivariant. The cross-section $\mathcal{K}^n$ is transversal to the orbits of G and the map ρ_G defined by

$$\rho_G(z^{(n)}) = \tilde{\rho}_B\left(z^{(n)}\right)\rho_A(z^{(n)}) = \rho_B\left(\rho_A(z^{(n)})\cdot z^{(n)}\right)\rho_A(z^{(n)}) \tag{11}$$

satisfy the condition $\rho_G(z^{(n)})\cdot z^{(n)} \in \mathcal{K}^n$, and hence is a moving frame for the G-action.

REMARK 3.6. We emphasize that G-equivariance of the map ρ_G, claimed above, follows from the correspondence between cross-sections to the orbits of G and G-equivariant maps from J^n to G, discussed on page 2. On the other hand, it can be established explicitly using B-equivariance of the map ρ_B and the following lemma.

LEMMA 3.7. *Let $G = A\cdot B$ act freely on a manifold N and let $\mathcal{K}_A$ be a cross-section for the action of A, invariant under the B-action. Then the map $\rho_A\colon N \to A \subset G$ defined by the condition $\rho_A(z)\cdot z \in \mathcal{K}_A$ is G-equivariant up to the action of B, that is, for any $g \in G$ there exists $b \in B$ such that*

$$\rho_A(g\cdot z) = b\rho_A(z)g^{-1}.$$

Proof. Let $z_1 = \rho_A(z)\cdot z$ and $z_2 = \rho_A(g\cdot z)g\cdot z$. By the definition of ρ_A, both z_1 and z_2 belong to $\mathcal{K}_A$ and hence from the freeness of the action it follows that $\rho_A(z_1) = \rho_A(z_2) = e \in G$. Let

$$h = \rho_A(g\cdot z)g\rho_A(z)^{-1} \in G, \tag{12}$$

then $z_2 = h\cdot z_1$. Since $G = A\cdot B$, there exist $a \in A$ and $b \in B$, such that $h = ab$. Then

$$e = \rho_A(z_2) = \rho_A(ab\cdot z_1) = \rho_A(b\cdot z_1)a^{-1}. \tag{13}$$

The last equality follows from A-equivariance of ρ_A. On the other hand, $b\cdot z_1 \in \mathcal{K}_A$, since $\mathcal{K}_A$ is invariant under the action of B, and thus $\rho_A(b\cdot z_1) = e$. We conclude from (13) that $a = e$ and hence $h = b$. The lemma now follows from (12). □

The cross-section $\mathcal{K}^n_A$ and $\mathcal{K}^n$ and the maps ρ_A and ρ_B can be extended to higher order jet bundles as it was done in Section 2. The non-constant coordinate functions of

$$\rho_G(z^{(k)})\cdot z^{(k)} = \rho_B\left(\rho_A(z^{(k)})\cdot z^{(k)}\right)\rho_A(z^{(k)})\cdot z^{(k)}, \quad k \geq n \tag{14}$$

provide a complete set of k-th order differential invariants for G.

REMARK 3.8. We notice that the coordinates of $\rho_A(z^{(k)}) \cdot z^{(k)}$ are invariant under the A-action and thus the formula above expresses the invariants of the G-action in terms of the invariants of its subgroup A.

We can summarize our construction in the following commutative diagram:

(15)

$$\begin{tikzcd} & & G \times J^\infty \arrow[d, shift left] \arrow[drr, "w_G"] & & \\ J^\infty \arrow[r, "\sigma_B"] \arrow[urr, "\sigma_G"] & B \times J^\infty \arrow[r, "\sigma_A"] & B \times A \times J^\infty \arrow[u, shift left] \arrow[r, "w_A"] & B \times J^\infty \arrow[r, "w_B"] & J^\infty \end{tikzcd}$$

where the maps w are defined by the prolonged group action:

$$\begin{aligned} w_A(b, a, z^{(\infty)}) &= (b, a \cdot z^{(\infty)}), \\ w_B(b, z^{(\infty)}) &= b \cdot z^{(\infty)}, \\ w_G(g, z^{(\infty)}) &= g \cdot z^{(\infty)} = w_B \circ w_A(b, a, z^{(\infty)}) \text{ where } g = ba. \end{aligned}$$

The maps σ are defined using the moving frames for A, B and G:

$$\begin{aligned} \sigma_A(b, z^{(\infty)}) &= (b, \rho_A(z^{(\infty)}), z^{(\infty)}), \\ \sigma_B(z^{(\infty)}) &= (\tilde{\rho}_B(z^{(\infty)}), z^{(\infty)}) = \left(\rho_B\left(\rho_A(z^{(\infty)}) \cdot z^{(\infty)}\right), z^{(\infty)}\right), \\ \sigma_G(z^{(\infty)}) &= (\rho_G(z^{(\infty)}), z^{(\infty)}) = \sigma_A \circ \sigma_B(z^{(\infty)}). \end{aligned}$$

We remind the reader that although all maps are written as if they were global, they might be defined only on open subsets of the manifolds appearing in (15). The manifolds $B \times A \times J^k$ and $G \times J^k$ are locally diffeomorphic, and this diffeomorphism is A-equivariant. The maps w_A and w_G are A-invariant, whence the maps σ_A and σ_B are A-equivariant, with respect to the A-actions on $B \times A \times J^\infty$, $B \times J^\infty$ and $G \times J^\infty$ defined respectively by:

$$\begin{aligned} \tilde{a} \cdot (b, a, z^{(\infty)}) &= (b, a\tilde{a}^{-1}, \tilde{a} \cdot z^{(\infty)}), \\ \tilde{a} \cdot (b, z^{(\infty)}) &= (b, \tilde{a} \cdot z^{(\infty)}), \\ \tilde{a} \cdot (g, z^{(\infty)}) &= (g\tilde{a}^{-1}, \tilde{a} \cdot z^{(\infty)}). \end{aligned}$$

We note that neither σ_A nor σ_B are B-equivariant, but their composition is. As it has been discussed in the previous section (see formula (5)) the forms

$$\omega_G^i = \sigma_G^* \, d_H \, w_G^*(x^i), \; i = 1, \ldots, p, \tag{16}$$

produce a contact G-invariant coframe on J^∞. Since A is a subgroup of G then the forms ω_G^i retain their invariant properties under the action of A. On the other hand, the moving frame ρ_A provide us with another horizontal coframe which is contact invariant under the action of A:

$$\omega_A^i = \sigma_A^* \, d_H \, w_A^*(x^i), \; i = 1, \ldots, p.$$

These two coframes are related by a linear transformation $w_G^i = \sum_{j=1}^p L_j^i w_A^j$, where L_j^i are functions on J^∞ invariant under the A-action. In fact, L_j^i can be explicitly expressed in terms of the fundamental invariants of A:

$$\omega_G^i = \sigma_B^* \, \sigma_A^* \, d_H \, w_A^* \, w_B^*(x^i) = \sigma_B^* \sigma_A^* \pi_H w_A^* d_H \chi^i(b_1, \ldots, b_l, x^1, \ldots, x^p, u_J^\alpha), \tag{17}$$

where $\chi^i = w_B^* x^i$ is a function on $B \times J^\infty$, written in local coordinates $b_1, \dots, b_l$, $x^1, \dots, x^p$, u_J^α. The forms $\sigma_A^* \,\pi_H\, w_A^*\, d_H \chi^i$ are obtained from $d_H \chi^i$ by replacing forms dx^j with ω_A^j and coordinate functions $x^1, , \dots, x^p, u_J^\alpha$ with their invariantizations $H^{(A)1}, , \dots, H^{(A)p}, I_J^{(A)\alpha}$. These forms provide a horizontal coframe on $B \times J^\infty$ which is contact invariant with respect to the action of A. The final pull-back σ_B^* is equivalent to the replacement of parameters $b_1, ..., b_l$ with the corresponding coordinates of $\rho_B(\rho_A(z^{(\infty)}) \cdot z^{(\infty)})$. The latter are expressed in terms of invariants of the A-action.

In many situations the following reformulation of Theorem 3.1 enables us to enlarge a moving frame for a transformation group A to a moving frame for a larger group containing A.

Theorem 3.9. *Let $\mathcal{O} \subset M$ be an orbit of G and let A be a subgroup which acts transitively on $\mathcal{O}$. Then $G = B \cdot A$, where B is the isotropy group of a point in $\mathcal{O}$. If in addition A acts locally freely on $\mathcal{O}$ then $A \cap B$ is discrete.*

Let n_A be the order of stabilization for A, then the action of A is (locally) free on a subset $\mathcal{V}_A \subset J^{n_A}(M,p)$. Assume that the action of A can be extended to the action of a group G containing A, so that there is a point $z_0 \in \mathcal{V}_A$ such that the orbits of A and G through z_0 coincide. If this is the case, then let B be the isotropy group of the point z_0. Due to the theorem above $G = B \cdot A$ and $A \cap B$ is discrete, and so Algorithm 3.5 can be applied. An especially favorable case is when the action of A on the regular set $\mathcal{V}_A \subset J^{n_A}(M,p)$ is transitive. Then a moving frame for A can be extended to a moving frame for any group G containing A.

4. Examples: Euclidean, Affine and Projective Actions on the Plane.

The group of Euclidean motions on the plane is a factor of the group of special affine motions. In turn, the group of special affine motions is a factor of the group of projective transformations on the plane. Applying the Inductive Algorithm 3.5 we express projective invariants in terms of affine, and affine invariants in terms of Euclidean. We also obtain the relations among the Euclidean, affine and projective arc-lengths, and the corresponding invariant differential operators.

Example 4.1. Let us use the moving frame for the special Euclidean group $SE(2,\mathbb{R})$ acting on curves in $\mathbb{R}^2$ obtained in Example 2.7 to build a moving frame for the special affine group. We recall that the moving frame for $SE(2,\mathbb{R})$ has been obtained on the first jet space by choosing the cross-section $\{x = 0, u = 0, u_x = 0\}$. The special Euclidean group acts transitively on $J^1(\mathbb{R}^2, 1)$ and the first invariant, the Euclidean curvature κ, appears on the second order of prolongation. The normalization of u_{xxx} and u_{xxxx} yields the third and the fourth order invariants $I_3^e = \kappa_s$ and $I_4^e = \kappa_{ss} + 3\kappa^3$.

The special affine transformation $SA(2,\mathbb{R})$ on the plane is the semi-direct product of the special linear group $SL(2,\mathbb{R})$ and translations in $\mathbb{R}^2$. We prolong it to the first jet bundle and notice that the isotropy group B of the point $z_0^{(1)} = \{x = 0, u = 0, u_x = 0\}$ is given by all linear transformations of the form

$$\begin{pmatrix} \tau & \lambda \\ 0 & \frac{1}{\tau} \end{pmatrix}.$$

Thus $SA(2,\mathbb{R}) = B \cdot SE(2,\mathbb{R})$ and $B \cap SE(2,\mathbb{R})$ is finite. In fact $B \cap SE(2,\mathbb{R}) = \{\mathrm{I}, -\mathrm{I}\}$. Now we prolong the action of B up to fourth order:

$$\begin{aligned}
x &\longrightarrow \tau x + \lambda u,\\
u &\longrightarrow \frac{1}{\tau} u,\\
u_x &\longrightarrow \frac{u_x}{\tau(\tau + \lambda u_x)},\\
u_{xx} &\longrightarrow \frac{u_{xx}}{(\tau + \lambda u_x)^3},\\
u_{xxx} &\longrightarrow \frac{(\tau + \lambda u_x) u_{xxx} - 3\lambda u_{xx}^2}{(\tau + \lambda u_x)^5},\\
u_{xxxx} &\longrightarrow \frac{(\tau + \lambda u_x)^2 u_{xxxx} - 10(\tau + \lambda u_x)\lambda u_{xx} u_{xxx} + 15\lambda^2 u_{xx}^3}{(\tau + \lambda u_x)^7}.
\end{aligned}$$

and restrict these transformations to the Euclidean cross-section $\mathcal{K}^4_E = \{z^{(4)} | \pi^4_1(z^{(4)}) = z^{(1)}_0\} = \{z^{(4)} | x = 0, u = 0, u_x = 0\}$, obtaining

$$\begin{aligned}
u_{xx} &\longrightarrow \frac{u_{xx}}{\tau^3},\\
u_{xxx} &\longrightarrow \frac{\tau u_{xxx} - 3\lambda u_{xx}^2}{\tau^5},\\
u_{xxxx} &\longrightarrow \frac{\tau^2 u_{xxxx} - 10\tau\lambda u_{xx} u_{xxx} + 15\lambda^2 u_{xx}^3}{\tau^7}.
\end{aligned}$$

The above action is free on the open subset $\{z^{(4)} \in \mathcal{K}^4_E | u_{xx} \neq 0\}$, where we choose the cross-section

$$\mathcal{K}^{(4)} = \{z^{(4)} \in \mathcal{K}^4_E | u_{xx} = 1, u_{xxx} = 0\}$$

to the orbits of B on $\mathcal{K}^4_E$. This produces a moving frame $\rho_B \colon \mathcal{K}^4_E \to B$:

$$\tau = (u_{xx})^{1/3} \text{ and } \lambda = \frac{u_{xxx}}{3(u_{xx})^{5/3}}.$$

The corresponding fourth order invariant for the action of B on $\mathcal{K}^4_E$ is

$$I^b_4 = \frac{u_{xx} u_{xxxx} - \frac{5}{3}(u_{xxx})^2}{(u_{xx})^{8/3}}. \tag{18}$$

We note that $\mathcal{K}^4$ can be viewed as a cross-section to the orbits of the entire group $SA(2,\mathbb{R})$ on the open subset of J^4 where $u_{xx} \neq 0$, and that due to formula (14) the forth order special affine invariant can be obtained by invariantization of I^b_4 with respect to Euclidean action, that is, by substitution of the normalized Euclidean invariants (9) into (18). Thus we obtain the expression of the lowest order special affine invariant μ in terms of Euclidean invariants:

$$\mu = I^a_4 = \frac{I^e_2 I^e_4 - \frac{5}{3}(I^e_3)^2}{(I^e_2)^{8/3}}. \tag{19}$$

One can rewrite the normalized Euclidean invariants in terms of the Euclidean curvature and its derivatives: $I^e_2 = \kappa$, $I^e_3 = \kappa_s$ and $I^e_4 = \kappa_{ss} + 3\kappa^3$, which leads to the expression:

$$\mu = \frac{\kappa(\kappa_{ss} + 3\kappa^3) - \frac{5}{3}\kappa_s^2}{\kappa^{8/3}}.$$

REMARK 4.2. The reader might notice that the affine invariant obtained above differs by a factor of 3 from the classical *affine curvature* (see, for instance, [**3**] p.14). The appearance of this factor can be predicted from the recurrence formulae (13.4) in [**9**].

In accordance with (11) the moving frame for the special affine group corresponding to the cross-section $\mathcal{K}^4$ is the product of two matrices:

$$\begin{pmatrix} \kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\ 0 & \frac{1}{\kappa^{1/3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Using formula (17), one can obtain an affine contact invariant horizontal form $d\alpha$ in terms of the Euclidean arc-length ds:

$$d\alpha = \sigma_B^*\,\sigma_E^*\,\pi_H\,w_E^*\,d_H\,w_B^*\,(x),$$

where the Euclidean invariantization of $d_H\,w_B^*\,(x) = (\tau + \lambda u_x)\,dx$ equals to $\tau\,ds$ and hence

$$d\alpha = \sigma_B^*(\tau\,ds) = (I_2^e)^{1/3}ds = \kappa^{1/3}ds. \tag{20}$$

The form $d\alpha$ is called the affine arc-length. Written in the standard coordinates $d\alpha = u_{xx}^{1/3}dx$. The relation (20) between the affine and the Euclidean arc-lengths provide a natural explanation for the affine curve evolution equation in [**19**]. The relation between invariant differential operators follows immediately:

$$\frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}}\frac{d}{ds},$$

which enables us to obtain all higher order affine invariants in terms of the Euclidean ones.

EXAMPLE 4.3. Let us now use the moving frame for the special affine group to build a moving frame for the projective group $PSL(3,\mathbb{R})$, whose local action on the plane is given by the transformations:

$$\begin{aligned} x &\mapsto \frac{\alpha x + \beta u + \gamma}{\delta x + \epsilon u + \zeta}, \\ u &\mapsto \frac{\lambda x + \nu u + \tau}{\delta x + \epsilon u + \zeta}, \end{aligned}$$

where the determinant of the corresponding 3×3 matrix equals to one. The affine moving frame above corresponds to the cross-section

$$z_0^{(3)} = \{x=0, u=0, u_1=0, u_2=1, u_3=0\} \in J^3.$$

The isotropy group B of $z_0^{(3)}$ for the prolonged action of $PSL(3,\mathbb{R})$ consists of the transformations:

$$\begin{pmatrix} 1 & ab & 0 \\ 0 & a & 0 \\ b & c & \frac{1}{a} \end{pmatrix}.$$

Thus $PSL(3,\mathbb{R}) = B\cdot SA(2,\mathbb{R})$ and $B\cap SA(2,\mathbb{R})$ is finite. The affine cross-section $\mathcal{K}_A^7 = \{z^{(7)}|\pi_3^7(z^{(7)}) = z_0^{(3)}\} = \{z^{(7)}|\, x=0, u=0, u_x=0, u_{xx}=1, u_{xxx}=0\}$ is invariant under the action of B. The seventh order prolongation of the B-action on

J^7 has been computed using the MAPLE package VESSIOT [**1**]. The restriction of this action to $\mathcal{K}^7_A$ is given by

$$\begin{aligned} u_4 &\to \frac{u_4 - 3a^2b^2 + 6ac}{a^2}, \\ u_5 &\to \frac{u_5}{a^3}, \\ u_6 &\to \frac{u_6 + 3abu_5 + 30u_4(2ac - a^2b^2) + 180a^2c(c - ab^2) + 45a^2b^2}{a^4}, \\ u_7 &\to \frac{u_7 + 7abu_6 + u_5(105ac - 42b^2a^2) - 35(u_4)^2ab}{a^5}. \end{aligned}$$

The above action is free on the open subset $\{z^{(7)} \in \mathcal{K}^7_A | u_5 \neq 0\}$, where we choose a cross-section

$$\mathcal{K}^7 = \{z^{(7)} \in \mathcal{K}^7_A | u_4 = 0, u_5 = 1, u_6 = 0\}$$

to the orbits of B on $\mathcal{K}^7_A$. This produces a moving frame $\rho_B \colon \mathcal{K}^7_A \to B$ given by

$$\begin{aligned} a &= (u_5)^{1/3}, \\ b &= \frac{5(u_4)^2 - u_6}{3(u_5)^{4/3}}, \\ c &= \frac{(u^6)^2 - 10u_6(u_4)^2 - 3u_4(u_5)^2 + 25(u_4)^4}{18(u_5)^{7/3}}. \end{aligned}$$

The corresponding seventh order differential invariant (for the action of B on $\mathcal{K}^7_A$) is

$$I^b_7 = \frac{6u_7u_5 - 7(u_6)^2 + 70(u_4)^2u_6 - 105u_4(u_5)^2 - 175(u_4)^4}{6(u_5)^{8/3}}. \tag{21}$$

We note that $\mathcal{K}^7$ can be viewed as a cross-section to the orbits of the entire group $PSL(3, \mathbb{R})$ on the open subset J^7, where $u_5 \neq 0$. Due to formula (14) the lowest order projective invariant can be obtained by invariantization of I^b_7 with respect to the affine action, that is, by substitution of the normalized affine invariants $I^a_4 = \mu$, $I^a_5 I^a_6$ and I^a_7 in (21). Note that we do not need the explicit formulae for these invariants. Thus we obtain a seventh order projective invariant η in terms of the special affine invariants:

$$\eta = I^p_7 = \frac{6I^a_7 I^a_5 - 7(I^a_6)^2 + 70(I^a_4)^2 I^a_6 - 105I^a_4(I^a_5)^2 - 175(I^a_4)^4}{6(I^a_5)^{8/3}}.$$

Using the recursion algorithm from [**9**] we can express the higher order affine invariants in terms of μ and its derivatives with respect to *affine* arc-length $d\alpha = u_{xx}^{1/3} dx$:

$$\begin{aligned} I^a_4 &= \mu, & I^a_5 &= \mu_\alpha, \\ I^a_6 &= \mu_{\alpha\alpha} + 5\mu^2, & I^a_7 &= \mu_{\alpha\alpha\alpha} + 17\mu\mu_\alpha. \end{aligned}$$

This leads to the formula:

$$\eta = \frac{-7\mu^2_{\alpha\alpha} + 6\mu_\alpha\mu_{\alpha\alpha\alpha} - 3\mu\mu^2_\alpha}{6\mu_\alpha^{8/3}}.$$

REMARK 4.4. The above expression can be compared with analogous formula (61) in [**7**]. Keeping in mind that the affine curvature used there differs from μ by a factor $\frac{1}{3}$, we notice that η differs from the classical projective curvature by a factor of $6^{-5/3}$.

The moving frame for the projective group is the product of the matrices:

$$\begin{pmatrix} 1 & -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_\alpha} & 0 \\ 0 & \mu_\alpha^{1/3} & 0 \\ -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_\alpha^{4/3}} & \frac{1}{18}\frac{\mu_{\alpha\alpha}^2-3\mu\mu_\alpha^2}{\mu_\alpha^{7/3}} & \frac{1}{\mu_\alpha^{1/3}} \end{pmatrix} \begin{pmatrix} \kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\ 0 & \frac{1}{\kappa^{1/3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

We can express the projective arc-length (that is, a horizontal form which is contact invariant with respect to the projective action) in terms of the affine arc-length $d\alpha$. We first lift the coordinate function x to $B \times J^\infty$ by $w_B^*(x) = \frac{x+abu}{bx+cu+\frac{1}{a}}$. The affine invariantization of $d_H\, w_B^*(x)$ produces a horizontal form $a\, d\alpha$ on $B \times J^\infty$, where $d\alpha$ is the affine arc-length (20). The projective arc-length equals to

$$d\varrho = \sigma_B^*\, a\, d\alpha = (I_5^a)^{1/3} d\alpha = \mu_\alpha^{1/3} d\alpha.$$

The relation between invariant derivatives, $\frac{d}{d\varrho} = \frac{1}{\mu_\alpha^{1/3}}\frac{d}{d\alpha}$, allows us to obtain all higher order projective invariants in terms of the affine ones.

Acknowledgements. I am grateful to Peter Olver and Mark Fels for sharing their vision of Cartan's moving frame method. I actively use and highly appreciate the software package VESSIOT, a powerful tool for computations on jet bundles, developed by Ian Anderson and his students. I would also like to thank the referee for the number of valuable comments, and Joshua Leslie and Thierry Robart for inviting me to the NSF-CBMS conference on "Geometry of Differential Equations" at Howard University.

References

[1] I. M. Anderson. *The Vessiot Handbook; Technical Report*. Utah Sate University, 2000.

[2] I. M. Anderson and Fels M. E. Symmetry reduction of variational bicomplexes and the principle of symmetric criticality. *American Journal of Mathematics*, 119:609–670, 1997.

[3] W. Blaschke. *Vorlesungen über Differentialgeometrie, vol II.* Verlag von Julius Springer, Berlin, 1923.

[4] É. Cartan. La méthode du repère mobile, la théorie des groupes continus, et les espaces généralisées. *Exposés de Géométrie*, 5:Hermann, Paris, 1935.

[5] É. Cartan. *Groupes finis et continus et la géométrie différentielle traitées par la methode du repère mobile*. Gauthier-Villars, Paris, 1937.

[6] É. Cartan. *Leçons sur la géométrie projective complexe*. Gauthier-Villars, Paris, 1950.

[7] O. Faugeras. Cartan's moving frame method and its application to the geometry and evolution of curves in the Euclidean, affine and projective planes. *Application of Invariance in Computer Vision, J.L Mundy, A. Zisserman, D. Forsyth (eds.) Springer-Verlag Lecture Notes in Computer Science*, 825:11–46, 1994.

[8] M. Fels and P. J. Olver. Moving Coframes. I. A Practical Algorithm. *Acta Appl. Math.*, 51:161–213, 1998.

[9] M. Fels and P. J. Olver. Moving Coframes. II. Regularization and Theoretical Foundations. *Acta Appl. Math.*, 55:127–208, 1999.

[10] R. B. Gardner. *The Method of Equivalence and its Applications*. SIAM, Philadelphia, 1989.

[11] V. V Gorbatsevich, A. L. Onishchik, and E. B. Vinberg. *Foundations of Lie Theory and Lie Transformations Groups*. Springer, 1993.

[12] M. L. Green. The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces. *Duke Math. Journal*, 45:735–779, 1978.

[13] P. A. Griffiths. On Cartan's method of Lie groups as applied to uniqueness and existence questions in differential geometry. *Duke Math. Journal*, 41:775–814, 1974.

[14] H. W. Guggenheimer. *Differential Geometry*. McGraw-Hill, New York, 1963.

[15] N. Kamran. Contributions to the study of the equivalence problem of Elie Cartan and its applications to partial and ordinary differential equations. *Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8º (2)*, 45(7), 1989.

[16] I. A. Kogan and P. J. Olver. Invariant Euler-Lagrange equations and the invariant variational bicomplex. *University of Minnesota, preprint*, 2000.

[17] P. J. Olver. *Equivalence Invariants and Symmetry*. Cambridge University Press, Cambridge, 1995.

[18] P. J. Olver. Joint invariant signatures. *Found. Comput. Math.*, 1:3–67, 2001.

[19] G. Sapiro and A. Tannenbaum. On affine plane curve evolution. *Journal of Functional Analysis*, 119:79–120, 1994.

Department of Mathematics, Yale University, New Haven, CT 06520
E-mail address: kogan@math.yale.edu