Corrections to

Kogan, I.A., and Olver, P.J., Invariant Euler-Lagrange equations and the invariant variational bicomplex, *Acta Appl. Math.* **76** (2003), 137–193.

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- Unfortunately, the variable κ is used to index the infinitesimal generators, Maurer–Cartan forms, etc., and also to denote curvature invariants, leading to notation clashes such as in equation (7.14) where it appears in both roles. To avoid this, the index κ should be everywhere changed to ℓ, which always runs from 1 to $r = \dim G$.

- Second paragraph of section 3: change $\mathrm{J}^n(m,p)$ to $\mathrm{J}^n(M,p)$.

- In the second equation in (5.25), the term $\varphi^\alpha_{K,\ell}$ is missing its second index, now denoted by ℓ:

$$\begin{aligned} d_{\mathcal{V}}\, H^i &= \iota\left(\sum_{\ell=1}^r \xi^i_\ell\,\beta^\ell\right) = \sum_{\ell=1}^r \Xi^i_\ell\,\varepsilon^\ell, \\ d_{\mathcal{V}}\, I^\alpha_K &= \iota\left(\theta^\alpha_K + \sum_{\ell=1}^r \varphi^\alpha_{K,\ell}\;\beta^\ell\right) = \vartheta^\alpha_K + \sum_{\ell=1}^r \Phi^\alpha_{K,\ell}\,\varepsilon^\ell. \end{aligned} \tag{5.25}$$

- In the second formula in (5.28), delete the summation over j:

$$d_{\mathcal{H}}\,\varpi^i = \sum_{j<k} Y^i_{jk}\,\varpi^j \wedge \varpi^k, \qquad \text{where} \qquad Y^i_{jk} = \sum_{\ell=1}^r \iota\left(A^\ell_j\, D_k\xi^i_\ell - A^\ell_k\, D_j\xi^i_\ell\right). \tag{5.28}$$

- page 164, third displayed formula: remove possibly confusing limits — the sum is over all multi-indices J:

$$\mathbf{E}_\alpha(L) = \sum_J (-D)_J\,\frac{\partial L}{\partial u^\alpha_J}$$

- In (7.14) and (7.15) the indices on Φ should both be subscripts to correspond to the notation used in (5.25). The summation index is now denoted by ℓ:

$$d_{\mathcal{V}}\,\kappa = \vartheta_{r-1} + \sum_{\ell=1}^r \Phi_{r-1,\ell}\,\varepsilon^\ell, \quad \text{where} \quad \varepsilon^\ell = \sum_j E^\ell_j\,\vartheta_j = \sum_j E^\ell_j\,\mathcal{F}_j(\vartheta) \equiv \mathcal{G}^\ell(\vartheta), \tag{7.14}$$

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta), \qquad \text{where} \qquad \mathcal{A} = \mathcal{F}_{r-1} + \sum_{\ell=1}^r \Phi_{r-1,\ell}\,\mathcal{G}^\ell \tag{7.15}$$

- In (7.16), the right hand side is missing a summation over what is now denoted by ℓ:

$$d_{\mathcal{V}}\,\varpi = \sum_{\ell=1}^{r} \left[\iota\left(\frac{\partial \xi_\ell}{\partial u}\right)\gamma^\ell \wedge \vartheta + \iota\,(D_x \xi_\ell)\;\varepsilon^\ell \wedge \varpi \right]. \tag{7.16}$$

- In (7.17), the d should be $d_{\mathcal{V}}$. Again, the summation index is now ℓ:

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi, \qquad \text{where} \qquad \mathcal{B} = \sum_{\ell=1}^{r} \left[\iota(D_x \xi_\ell)\,\mathcal{G}^\ell - \iota\left(\frac{\partial \xi_\ell}{\partial u}\right) C^\ell \right] \tag{7.17}$$

- On page 174 in the second-to-last displayed formula, the right hand side is missing a minus sign:

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta^u \wedge \varpi$$

- On page 174 in the next-to-last displayed formula, both expressions are missing minus signs:

$$\mathcal{B} = \big(-\,\kappa, 0\big) \qquad \text{so that} \qquad \mathcal{B}^* = \begin{pmatrix} -\,\kappa \\ 0 \end{pmatrix}.$$

- In (9.11), the left hand side is missing a minus sign:

$$-\,F\,d_{\mathcal{H}}\,\sigma \wedge \boldsymbol{\varpi}_{(j)} \equiv (\mathcal{D}_j^\dagger F)\,\sigma \wedge \boldsymbol{\varpi}. \tag{9.11}$$

- In (9.13), the $=$ should be $\equiv$:

$$F(\mathcal{D}_j \psi) \wedge \boldsymbol{\varpi} \equiv -\,(\mathcal{D}_j + Z_j) F\,\psi \wedge \boldsymbol{\varpi} = (\mathcal{D}_j^\dagger F)\,\psi \wedge \boldsymbol{\varpi} \tag{9.13}$$

- In (9.20), the second formula is missing a summation over i:

$$d_{\mathcal{V}}\,I^\alpha = \sum_{\beta=1}^{q} \mathcal{A}^\alpha_\beta(\vartheta^\beta), \qquad\qquad d_{\mathcal{V}}\,\varpi^j = \sum_{i=1}^{p}\sum_{\beta=1}^{q} \mathcal{B}^j_{i,\beta}(\vartheta^\beta) \wedge \varpi^i, \tag{9.20}$$

- In (9.34), the Y's in the second pair of formulas should be reversed:

$$\begin{aligned} d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(1)} &= d_{\mathcal{H}}\,\varpi^2 = -\,\frac{I_{12}}{I}\,\boldsymbol{\varpi}, \\ d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(2)} &= -\,d_{\mathcal{H}}\,\varpi^1 = \frac{I_{11}}{I}\,\boldsymbol{\varpi}, \end{aligned} \qquad \text{so} \qquad \begin{aligned} Z_1 &= Y^2_{12} = -\,\frac{I_{12}}{I}\,, \\ Z_2 &= -Y^1_{12} = \frac{I_{11}}{I}\,. \end{aligned} \tag{9.34}$$

- In the published paper, a sign error in equation (9.43) propagated, affecting the subsequent displayed equation, equations (9.45), (9.46), and particularly (9.47). The corrected version of the affected text follows:

On the other hand,

$$\begin{aligned} d_{\mathcal{V}}\, \varpi^1 &= -\,\kappa^1\, \vartheta \wedge \varpi^1 + \frac{1}{\kappa^1 - \kappa^2}\big(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1 \,\big) \vartheta \wedge \varpi^2, \\ d_{\mathcal{V}}\, \varpi^2 &= \frac{1}{\kappa^2 - \kappa^1}\big(\, \mathcal{D}_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2 \,\big) \vartheta \wedge \varpi^1 - \kappa^2\, \vartheta \wedge \varpi^2, \end{aligned} \tag{9.43}$$

which yields the Hamiltonian operator complex

$$\begin{aligned} \mathcal{B}_1^1 &= -\,\kappa^1, \\ \mathcal{B}_2^2 &= -\,\kappa^2, \end{aligned} \qquad \mathcal{B}_2^1 = \frac{1}{\kappa^1 - \kappa^2}\big(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1 \,\big) = \frac{1}{\kappa^1 - \kappa^2}\big(\, \mathcal{D}_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2 \,\big) = -\,\mathcal{B}_1^2,$$

the equality following from the commutation formula (9.35). Therefore, according to our fundamental formula (9.24), the Euler-Lagrange equations for a Euclidean-invariant variational problem (9.40) are

$$\begin{aligned} 0 = \mathbf{E}(L) = {} & \big[\, (\mathcal{D}_1 + Z_1)^2 - (\mathcal{D}_2 + Z_2) \cdot Z_2 + (\kappa^1)^2 \,\big]\, \mathcal{E}_1(\widetilde{L}) \\ & + \big[\, (\mathcal{D}_2 + Z_2)^2 - (\mathcal{D}_1 + Z_1) \cdot Z_1 + (\kappa^2)^2 \,\big]\, \mathcal{E}_2(\widetilde{L}) + \kappa^1\, \mathcal{H}_1^1(\widetilde{L}) + \kappa^2\, \mathcal{H}_2^2(\widetilde{L}) \\ & + \big[\, (\mathcal{D}_2 + Z_2)(\mathcal{D}_1 + Z_1) + (\mathcal{D}_1 + Z_1) \cdot Z_2 \,\big] \cdot \left(\frac{\mathcal{H}_2^1(\widetilde{L}) - \mathcal{H}_1^2(\widetilde{L})}{\kappa^1 - \kappa^2} \right). \end{aligned} \tag{9.44}$$

As before, $\mathcal{E}_\alpha(\widetilde{L})$ are the invariant Eulerians with respect to the principal curvatures κ^α, while $\mathcal{H}_j^i(\widetilde{L})$ are the invariant Hamiltonians based on (9.41).

In particular, if $\widetilde{L}(\kappa^1, \kappa^2)$ does not depend on any differentiated invariants, (9.44) reduces to

$$\mathbf{E}(L) = \big[\, (\mathcal{D}_1^\dagger)^2 + \mathcal{D}_2^\dagger \cdot Z_2 + (\kappa^1)^2 \,\big]\, \frac{\partial \widetilde{L}}{\partial \kappa^1} + \big[\, (\mathcal{D}_2^\dagger)^2 + \mathcal{D}_1^\dagger \cdot Z_1 + (\kappa^2)^2 \,\big]\, \frac{\partial \widetilde{L}}{\partial \kappa^2} - (\kappa^1 + \kappa^2) \widetilde{L}. \tag{9.45}$$

For example, the problem of minimizing surface area has invariant Lagrangian $\widetilde{L} = 1$, and so (9.45) gives the Euler-Lagrange equation

$$\mathbf{E}(L) = -\,(\kappa^1 + \kappa^2) = -\,2H = 0, \tag{9.46}$$

and so we conclude that minimal surfaces have vanishing mean curvature. As noted above, the Gauss–Bonnet Lagrangian $\widetilde{L} = K = \kappa^1 \kappa^2$ is an invariant divergence, and hence its the Euler-Lagrange equation is identically zero. The mean curvature Lagrangian $\widetilde{L} = H = \frac{1}{2}(\kappa^1 + \kappa^2)$ has Euler-Lagrange equation

$$\tfrac{1}{2}\big[\, (\kappa^1)^2 + (\kappa^2)^2 - (\kappa^1 + \kappa^2)^2 \,\big] = -\,\kappa^1\, \kappa^2 = -K = 0. \tag{9.47}$$

For the Willmore Lagrangian $\widetilde{L} = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$, [**3**, **6**], formula (9.44) immediately gives the known Euler-Lagrange equation

$$\mathbf{E}(L) = \Delta(\kappa^1 + \kappa^2) + \tfrac{1}{2}(\kappa^1 + \kappa^2)(\kappa^1 - \kappa^2)^2 = 2\,\Delta H + 4(H^2 - K)H = 0, \qquad (9.48)$$

where

$$\Delta = (\mathcal{D}_1 + Z_1)\mathcal{D}_1 + (\mathcal{D}_2 + Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^\dagger \cdot \mathcal{D}_1 - \mathcal{D}_2^\dagger \cdot \mathcal{D}_2 \qquad (9.49)$$

is the Laplace–Beltrami operator on our surface.

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