Corrections to

Kogan, I.A., and Olver, P.J., Invariant Euler-Lagrange equations and the invariant variational bicomplex, *Acta Appl. Math.* **76** (2003), 137–193.

Last updated: November 3, 2011.

Second paragraph of section 3: change $\mathrm{J}^n(m,p)$ to $\mathrm{J}^n(M,p)$.

In (7.16), the right hand side is missing a summation over κ:

$$d_{\mathcal{V}}\,\varpi = \sum_{\kappa=1}^{r}\left[\iota\left(\frac{\partial\xi_\kappa}{\partial u}\right)\gamma^\kappa\wedge\vartheta + \iota\,(D_x\xi_\kappa)\;\varepsilon^\kappa\wedge\varpi\right]. \tag{7.16}$$

In (7.17), the d should be $d_{\mathcal{V}}$:

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta)\wedge\varpi, \qquad \text{where} \qquad \mathcal{B} = \sum_{\kappa=1}^{r}\left[\iota(D_x\xi_\kappa)\,\mathcal{G}^\kappa - \iota\left(\frac{\partial\xi_\kappa}{\partial u}\right)C^\kappa\right] \tag{7.17}$$

On page 174 in the next-to-last displayed formula, both expressions are missing minus signs:

$$\mathcal{B} = \big(-\kappa, 0\big) \qquad \text{so that} \qquad \mathcal{B}^* = \begin{pmatrix} -\kappa \\ 0 \end{pmatrix}.$$

In (9.11), the left hand side is missing a minus sign:

$$-\,F\,d_{\mathcal{H}}\,\sigma\wedge\boldsymbol{\varpi}_{(j)} \equiv (\mathcal{D}_j^\dagger F)\,\sigma\wedge\boldsymbol{\varpi}. \tag{9.11}$$

In (9.13), the $=$ should be $\equiv$:

$$F(\mathcal{D}_j\psi)\wedge\boldsymbol{\varpi} \equiv -\,(\mathcal{D}_j + Z_j)F\,\psi\wedge\boldsymbol{\varpi} = (\mathcal{D}_j^\dagger F)\,\psi\wedge\boldsymbol{\varpi} \tag{9.13}$$

In (9.20), the second formula is missing a summation over i:

$$d_{\mathcal{V}}\,I^\alpha = \sum_{\beta=1}^{q}\mathcal{A}_\beta^\alpha(\vartheta^\beta), \qquad d_{\mathcal{V}}\,\varpi^j = \sum_{i=1}^{p}\sum_{\beta=1}^{q}\mathcal{B}_{i,\beta}^j(\vartheta^\beta)\wedge\varpi^i, \tag{9.20}$$

In (9.34), the Y's in the second pair of formulas should be reversed:

$$\begin{aligned} d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(1)} &= d_{\mathcal{H}}\,\varpi^2 = -\,\frac{I_{12}}{I}\,\boldsymbol{\varpi}, \\ d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(2)} &= -\,d_{\mathcal{H}}\,\varpi^1 = \frac{I_{11}}{I}\,\boldsymbol{\varpi}, \end{aligned} \qquad \text{so} \qquad \begin{aligned} Z_1 &= Y_{12}^2 = -\,\frac{I_{12}}{I}\,, \\ Z_2 &= -Y_{12}^1 = \frac{I_{11}}{I}\,. \end{aligned} \tag{9.34}$$

• thanks to Francis Valiquette for spotting many of these typos.

In the published paper, a sign error in equation (9.43) propagated, affecting the subsequent displayed equation, equations (9.45), (9.46), and particularly (9.47).

The corrected version of the affected text follows:

On the other hand,

$$
\begin{aligned}
d_{\mathcal{V}}\,\varpi^1 &= -\,\kappa^1\,\vartheta\wedge\varpi^1 + \frac{1}{\kappa^1-\kappa^2}\,(\,\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\,)\vartheta\wedge\varpi^2,\\
d_{\mathcal{V}}\,\varpi^2 &= \frac{1}{\kappa^2-\kappa^1}\,(\,\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\,)\vartheta\wedge\varpi^1 - \kappa^2\,\vartheta\wedge\varpi^2,
\end{aligned}
\tag{9.43}
$$

which yields the Hamiltonian operator complex

$$
\begin{aligned}
\mathcal{B}^1_1 &= -\,\kappa^1,\\
\mathcal{B}^2_2 &= -\,\kappa^2,
\end{aligned}
\qquad
\mathcal{B}^1_2 = \frac{1}{\kappa^1-\kappa^2}\,(\,\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\,) = \frac{1}{\kappa^1-\kappa^2}\,(\,\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\,) = -\,\mathcal{B}^2_1,
$$

the equality following from the commutation formula (9.35). Therefore, according to our fundamental formula (9.24), the Euler-Lagrange equations for a Euclidean-invariant variational problem (9.40) are

$$
\begin{aligned}
0 = \mathbf{E}(L) = {}& \big[\,(\mathcal{D}_1+Z_1)^2 - (\mathcal{D}_2+Z_2)\cdot Z_2 + (\kappa^1)^2\,\big]\,\mathcal{E}_1(\widetilde{L})\\
&+ \big[\,(\mathcal{D}_2+Z_2)^2 - (\mathcal{D}_1+Z_1)\cdot Z_1 + (\kappa^2)^2\,\big]\,\mathcal{E}_2(\widetilde{L}) + \kappa^1\,\mathcal{H}^1_1(\widetilde{L}) + \kappa^2\,\mathcal{H}^2_2(\widetilde{L})\\
&+ \big[\,(\mathcal{D}_2+Z_2)(\mathcal{D}_1+Z_1) + (\mathcal{D}_1+Z_1)\cdot Z_2\,\big]\cdot\left(\frac{\mathcal{H}^1_2(\widetilde{L}) - \mathcal{H}^2_1(\widetilde{L})}{\kappa^1-\kappa^2}\right).
\end{aligned}
\tag{9.44}
$$

As before, $\mathcal{E}_\alpha(\widetilde{L})$ are the invariant Eulerians with respect to the principal curvatures κ^α, while $\mathcal{H}^i_j(\widetilde{L})$ are the invariant Hamiltonians based on (9.41).

In particular, if $\widetilde{L}(\kappa^1,\kappa^2)$ does not depend on any differentiated invariants, (9.44) reduces to

$$
\mathbf{E}(L) = \big[\,(\mathcal{D}_1^\dagger)^2 + \mathcal{D}_2^\dagger\cdot Z_2 + (\kappa^1)^2\,\big]\,\frac{\partial\widetilde{L}}{\partial\kappa^1} + \big[\,(\mathcal{D}_2^\dagger)^2 + \mathcal{D}_1^\dagger\cdot Z_1 + (\kappa^2)^2\,\big]\,\frac{\partial\widetilde{L}}{\partial\kappa^2} - (\kappa^1+\kappa^2)\widetilde{L}.
\tag{9.45}
$$

For example, the problem of minimizing surface area has invariant Lagrangian $\widetilde{L}=1$, and so (9.45) gives the Euler-Lagrange equation

$$
\mathbf{E}(L) = -\,(\kappa^1+\kappa^2) = -\,2H = 0,
\tag{9.46}
$$

and so we conclude that minimal surfaces have vanishing mean curvature. As noted above, the Gauss–Bonnet Lagrangian $\widetilde{L} = K = \kappa^1\kappa^2$ is an invariant divergence, and hence its the Euler-Lagrange equation is identically zero. The mean curvature Lagrangian $\widetilde{L} = H = \frac{1}{2}(\kappa^1+\kappa^2)$ has Euler-Lagrange equation

$$
\tfrac{1}{2}\big[\,(\kappa^1)^2 + (\kappa^2)^2 - (\kappa^1+\kappa^2)^2\,\big] = -\,\kappa^1\,\kappa^2 = -K = 0.
\tag{9.47}
$$

For the Willmore Lagrangian $\widetilde{L} = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$, [**3**, **6**], formula (9.44) immediately gives the known Euler-Lagrange equation

$$
\mathbf{E}(L) = \Delta(\kappa^1+\kappa^2) + \tfrac{1}{2}(\kappa^1+\kappa^2)(\kappa^1-\kappa^2)^2 = 2\,\Delta H + 4(H^2-K)H = 0,
\tag{9.48}
$$

where

$$
\Delta = (\mathcal{D}_1+Z_1)\mathcal{D}_1 + (\mathcal{D}_2+Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^\dagger\cdot\mathcal{D}_1 - \mathcal{D}_2^\dagger\cdot\mathcal{D}_2
\tag{9.49}
$$

is the Laplace–Beltrami operator on our surface.