Integral Invariants for 3D Curves: An Inductive Approach

Shuo Feng[*], Irina A. Kogan[**], Hamid Krim[*]

[*] ECE Department, North Carolina State University, Raleigh, NC, 27695-7914

[**] Department of Mathematics, North Carolina State University, Raleigh, NC, 27695-8205

ABSTRACT

In this paper we obtain, for the first time, explicit formulae for integral invariants for curves in 3D with respect to the special and the full affine groups. Using an inductive approach we first compute Euclidean integral invariants and use them to build the affine invariants. The motivation comes from problems in computer vision. Since integration diminishes the effects of noise, integral invariants have advantage in such applications. We use integral invariants to construct signatures that characterize curves up to the special affine transformations.

Keywords: 3D affine and Euclidean transformations, integral affine invariants, moving frames, object classification

1. INTRODUCTION

Invariants under the actions of the Euclidean, affine and projective groups are widely used in shape/object recognition problems in image processing and computer vision.[1–5] Differential invariants, such as Euclidean curvature and torsion for space curves, are the most classical. The affine and projective counterparts of curvature and torsion may also be defined. The practical utilization of differential invariants is, however, limited due to their high sensitivity to noise. Indeed, Euclidean curvature and torsion depend on derivatives of up to order 2 and 3 respectively, and their affine analogs depend on derivatives of up to order 6. If the original data is noisy, the numerical differentiation amplifies the effects of noise. This motivated the high interest in other types of invariants such as semi-differential, or joint invariants[6–9] and various types of integral invariants.[10–13] Integral invariants in the above references depend on quantities obtained by integration of various functions along a curve. The type of integral invariants that we consider was introduced by,[13] and can be thought as the 1-dimensional analog of moment invariants.[14, 15] Since integration reduces the effect of noise, integral invariants hold a clear advantage in practical applications. Explicit expressions for integral invariants, however, appear to be known only for curves in 2D, as computations become challenging in 3D.

In this paper we obtain, for the first time, explicit formulae of integral affine invariants for 3D curves. The standard action of the affine group on $\mathbb{R}^3$ induces an action on curves. Following the approach of Ref. 13 we prolong the action to certain integral expressions, called potentials, and then compute invariants that depend on these integral variables. The computation was performed using an inductive variation[16] of the Fels-Olver moving frame construction.[17] This approach allows us to build invariants for the entire group from invariants of its subgroup, and in our case affine invariants in terms of Euclidean ones. We begin by constructing the affine invariant for plane curves (first obtained in Ref. 13) in terms of Euclidean invariants, and then obtain two new special affine invariants for space curves. Their quotient is invariant under the action of the full affine group. The result is given in Section 3, while the inductive derivation is postponed to Section 4. In Section 3 we use the derived invariants to define integral signatures. The signature of a curve is obtained by plotting one independent invariant, evaluated on the curve, versus another. If one curve can be mapped to another curve by a group transformation, then their signatures coincide. The signatures are independent of a parameterization and of a choice of the initial point on a curve. Applications of signatures based on differential invariants and on joint invariants were, for instance, explored in curve classification in Ref. 8, 9, 18–20. In Ref. 12 integral signatures for *plane curves* were constructed to classify curves up to Euclidean transformations. These signatures are based on a different type of integral invariants than the ones considered in this paper. A detailed analysis of the signatures constructed in the present paper and practical experiments are in progress.[?]

Further author information: (Send correspondence to I. A. Kogan)

I.A.K.: E-mail: iakogan@ncsu.edu, Telephone: 1 919-513-7437

S.F.: E-mail: sfeng2@ncsu.edu, Telephone: 1 919-515-7807

H.K.: ahk@ncsu.edu, Telephone: 1 919-513-2270

2. GROUP ACTION AND INVARIANTS

2.1. Definitions

Definition 2.1. An action of a group G on a set S is a map $\alpha: G \times S \to S$ that satisfies the following two properties:

1. $\alpha(e, s) = s$, $\forall s \in S$, where e is the identity of the group.
2. $\alpha(g_1, \alpha(g_2, s)) = \alpha(g_1\, g_2, s)$, for all $s \in S$ and $g_1, g_2 \in G$.

For $g \in G$ and $s \in S$ we write $\alpha(g, s) = g \cdot s = \overline{s}$.

We will use the following terminology from group theory.

Definition 2.2. The *orbit* of a point $s \in S$ is the set $O_s = \{g \cdot s | g \in G\}$.

Definition 2.3. A subset $S_1 \subset S$ is *invariant* if $g \cdot s \in S_1$ for $\forall s \in S_1$ and $\forall g \in G$.*

Definition 2.4. An action of G is called *free* if $\forall s \in S$ the isotropy group $G_s = \{g \in G | g \cdot s = s\} = \{e\}$. An action of G is called *locally free* if $\forall s \in S$ the isotropy group G_s is discrete.

Let $GL(n)$ denote a group of non-degenerate $n \times n$ matrices. Its subgroup of matrices with determinant 1 is denoted by $SL(n)$. The *orthogonal group* is $O(n) = \{A \in GL(n) | AA^T = I\}$, while the *special orthogonal group* is $SO(n) = \{A \in O(3) | \det A = 1\}$. The semi-direct product of $GL(n)$ and $\mathbb{R}^n$ is called the *affine group*: $A(n) = GL(n) \ltimes \mathbb{R}^n$. Its subgroup $SA(n) = SL(n) \ltimes \mathbb{R}^n$ is called the *special affine group.* The *Euclidean group* is $E(n) = O(n) \ltimes \mathbb{R}^n$. Its subgroup $SE(n) = SO(n) \ltimes \mathbb{R}^n$ is called the *special Euclidean group.*

In the paper we consider the action of the affine group $A(3)$ and its subgroups on $\mathbb{R}^3$ by a composition of a linear transformation and a translation:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{1}$$

Definition 2.5. A function $f: S \to \mathbb{R}$ is called *invariant* if

$$f(g \cdot s) = f(s),\ \forall g \in G \text{ and } \forall s \in S \tag{2}$$

2.2. Prolongation of a group action

A group action (1) on $\mathbb{R}^3$ induces an action on curves $\gamma(t) = (x(t), y(t), z(t)) \to \overline{\gamma}(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t))$. Our goal is to obtain invariants that classify curves up to the affine transformation. The classical method of obtaining such invariants is to prolong the action to the set of derivatives $\{x_i, y_i, z_i | i = 1..l\}$ of a sufficiently high order:[21]

$$\overline{x_1}(t) = \frac{d\overline{x}(t)}{dt}, \quad \overline{x_{i+1}}(t) = \frac{d\overline{x_i}(t)}{dt} \text{ and similarly for } y_i \text{ and } z_i \text{ for } i > 0. \tag{3}$$

Definition 2.6. Functions of $\{x, y, z, x_i, y_i, z_i \,|\, i = 1..l\}$ that are invariant under the prolonged action (3) are called *differential invariants* of order l.

For Euclidean and affine groups, as well as projective groups, acting on curves in 3D, the two lowest order invariants are called curvature and torsion, and are classically known in differential geometry.

*An orbit is a smallest invariant subset. Any invariants subset is the union of orbits.

As noted in the introduction, differential invariants are highly sensitive to noise. Extending the approach of Ref. 13 to curves in 3D, entails our prolonging the action (1) to $N = \frac{l(l+1)(l+5)}{2}$ integral variables of order l.

$$\begin{aligned}
X_{ijk}(t) &= \int_0^t x^i y^j z^k dx, \quad j+k \neq 0 \\
Y_{ijk}(t) &= \int_0^t x^i y^j z^k dy, \quad i+k \neq 0 \\
Z_{ijk}(t) &= \int_0^t x^i y^j z^k dz, \quad i+j \neq 0
\end{aligned} \tag{4}$$

where the integrals are taken along the curve $\gamma(t)$ and $i+j+k = l$.

Let $\gamma(0) = (x(0), y(0), z(0))^T$, $A \in GL(3)$, $v \in R^3$. Following the argument of Proposition 1 of Ref. 13 one can show that the transformations on $\mathbb{R}^{N+6}$

$$\begin{aligned}
\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} &= A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \begin{pmatrix} \overline{x}(0) \\ \overline{y}(0) \\ \overline{z}(0) \end{pmatrix} = A \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
\overline{X_{ijk}} &= \int_0^t \overline{x}^i \overline{y}^j \overline{z}^k d\overline{x} \text{ and similarly for } \overline{Y_{ijk}} \text{ and } \overline{Z_{ijk}}
\end{aligned} \tag{5}$$

induced by (1) satisfy the axioms of a group action, and respect the relations among variables $x, y, z, x(0), y(0), z(0), X_{ijk}, Y_{ijk}, Z_{ijk}$ that follow from the integration by parts formula. For instance $X_{010} = x(t)y(t) - x(0)y(0) - Y_{100}$ and $\overline{X_{0,1,0}} = \overline{x}(t)\overline{y}(t) - \overline{x}(0)\overline{y}(0) - \overline{Y_{1,0,0}}$. Thus the action (5) restricts to the subvariety of $\mathbb{R}^{N+6}$ defined by these relations.

Definition 2.7. Functions of $\{x, y, z, x(0), y(0), z(0), X_{ijk}, Y_{ijk}, Z_{ijk} \,|\, i+j+k = l\}$ that are invariant under the action (5) are called *integral invariants* of order l.

We reduce the problem of finding invariants under the action (5) to an equivalent but simpler problem of finding invariant functions of variables $\{X, Y, Z, X_{i,j,k}, Y_{i,j,k}, Z_{i,j,k} \,|\, i+j+k = l\}$ under the action of $GL(3)$ defined by

$$\begin{aligned}
\begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{pmatrix} &= A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \\
\overline{X_{ijk}} &= \int_0^t \overline{X}^i \overline{Y}^j \overline{Z}^k d\overline{X} \text{ and similarly for } \overline{Y_{ijk}} \text{ and } \overline{Z_{ijk}}
\end{aligned} \tag{6}$$

by introducing new variables

$$X = x - x(0), Y = y - y(0), Z = z - z(0) \tag{7}$$

and making the corresponding substitution in the integrals. Invariants with respect to (5) may be obtained from invariants with respect to (6) by making substitution (7).†

3. INTEGRAL SIGNATURES

In Section 4 we obtain the following two second order integral invariants, which we use to classify curves in 3D with respect to the special affine transformations.

$$\begin{aligned}
I_1 &= n_1 X + n_2 Z - n_3 Y \\
I_2 &= 2\, n_1 (XYZ^2 - 3\, Z_{011} X + 3\, Y Z_{101} - Z Z_{110} - 2\, Z Y_{101}) + n_2 (2\, XY^2 Z + 3\, X Z_{020} \\
&- 6\, Z X_{020} - 4\, Y Z_{110} - 2\, Y Y_{101}) - 2\, n_3 (3\, Y X_{101} - 3\, Z X_{110} + X Z_{110} - X Y_{101})
\end{aligned} \tag{8}$$

†This reduction by the group of translations can be put in the context of inductive method described in Section 4. We feel, however, that making this step "upfront" makes the presentation more transparent.

where

$$n_1 = YZ - 2\,Z_{010}, \quad n_2 = XY - 2\,Y_{100}, \text{ and } n_3 = XZ - 2\,Z_{100}$$

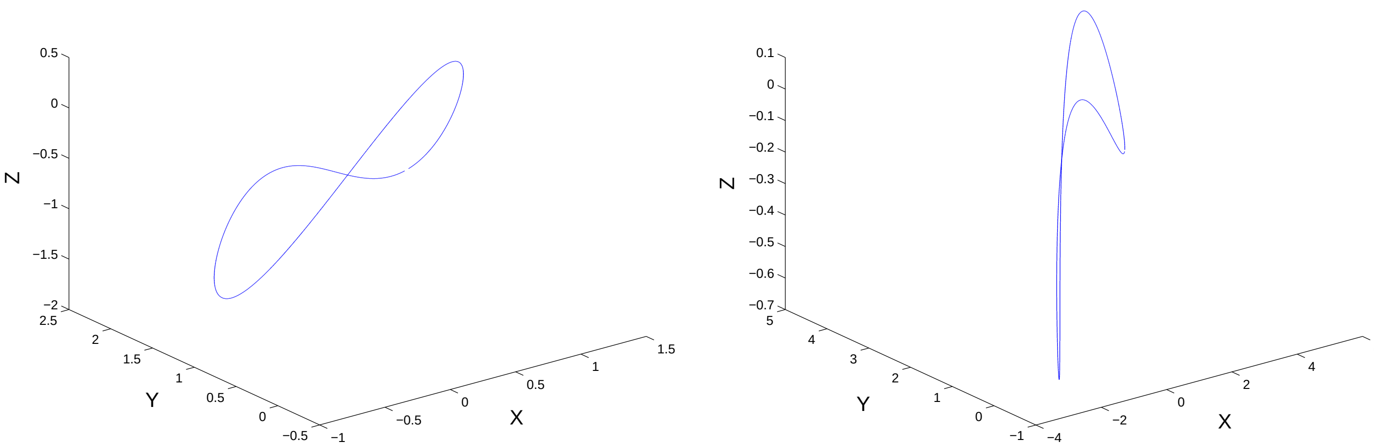
represent certain areas.

The *signature* of a curve $\gamma(t)$ is obtained by first evaluating I_1 and I_2 on this curve, and by then plotting a parameterized curve $(I_1(t), I_2(t))$ in $\mathbb{R}^2$.

Example 3.1. The signature of a space curve

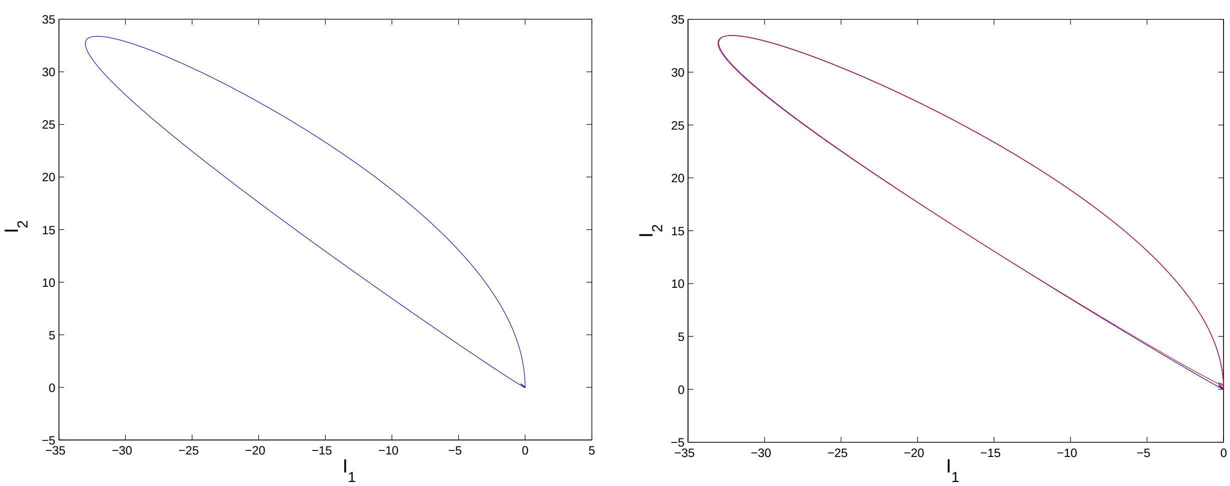
$$\gamma(t) = (\sin t - 1/5\,\cos^2 t + 1/5,\ 1/2\,\sin t - \cos t + 1\,\sin^2 t + \cos t - 1),$$

shown in Fig. 1-a is a plane curve shown on Fig 2-a.



(a) (b)

Figure 1. (a) original curve $\gamma(t)$ (b) transformed curved $\overline{\gamma(t)}$

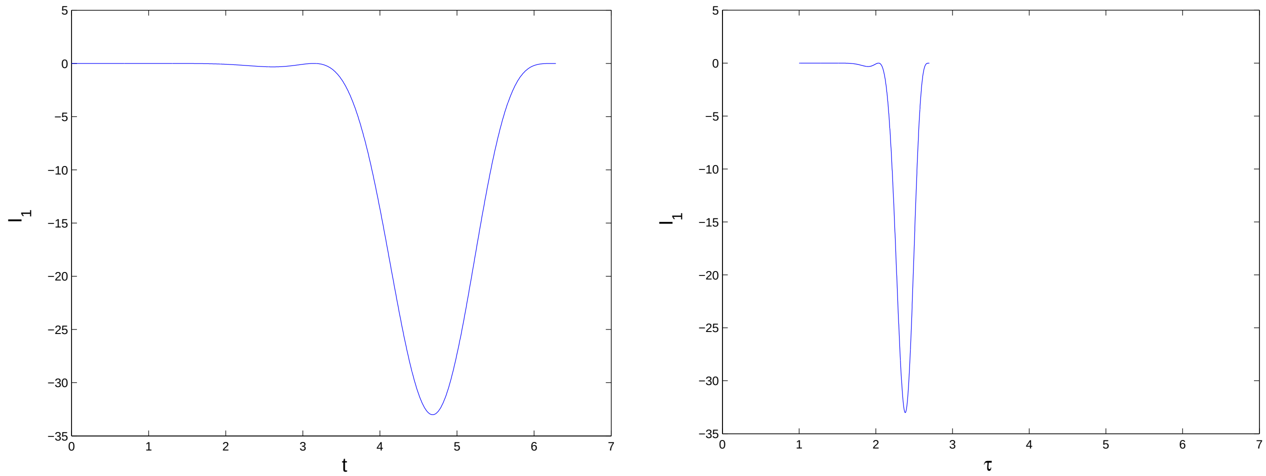


(a) (b)

Figure 2. (a) Signatures for $\gamma(t)$ and $\overline{\gamma(t)}$ (b) Their numerical approximations for discretizations of $\gamma(t)$ and $\overline{\gamma(t)}$

A curve $\overline{\gamma}(t)$ is obtained from $\gamma(t)$ by an affine transformation. As illustrated in Fig 2-a, their signatures coincide. Fig 2-a shows that the numerical approximations of signatures for discretizations of $\gamma(t)$ and $\overline{\gamma(t)}$ are very close.

The advantage of signatures is that they depend on neither the parameterization nor on the initial point. This is in contrast to plotting invariants with respect to a parameter which is dependent on the choice of parameterization as illustrated in Fig 3.[‡]



(a) (b)

Figure 3. (a) Invariant I_1 for $\gamma(t)$ (b) I_1 for reparameterization of γ with $\tau = \sqrt{t+1}$

A detailed analysis of signatures proposed in this paper is in progress. In particular, we conjecture that, similarly to the case of differential signatures, integral signatures defined in this paper distinguish equivalence classes of curves, and that information about the symmetries of the curve can be extracted from the signatures. Having in mind applications to object recognition, we are performing a quantitative analysis of their sensitivity to noise in comparison to signatures based on other types of invariants.[?]

4. DERIVATION OF INVARIANTS

4.1. Cross-section and moving frame map

Building on the works Ref. 22–24, Fels and Olver[17] generalized Cartan's normalization procedure,[25] and proposed a general algorithm for computing invariants. The Fels-Olver algorithm relies on a map $\rho: S \to G$ with an *equivariant* property:

$$\rho(g \cdot s) = \rho(s) \cdot g^{-1}, \forall g \in G, \forall s \in S. \tag{9}$$

From Theorem 4.4 in Ref. 17, it follows that such map exists if and only if the action of G is free and, in addition, there exists a global *cross-section*, i.e a subset $\mathcal{K} \subset S$ that intersects each orbit O_s at a unique point. Indeed, under the above assumption the map ρ may be defined by the condition $\rho(s) \cdot s \in \mathcal{K}$. Then $\rho(s) \cdot s = \rho(g \cdot s) \cdot (g \cdot s)$ is the unique point of the intersection of O_s and $\mathcal{K}$. From the freeness it follows that s may be "cancelled" and hence the condition (9) is satisfied.

If G is a Lie group acting smoothly on $\mathbb{R}^n$ and both $S \subset \mathbb{R}^n$ and $\mathcal{K} \subset S$ are smooth submanifolds, then $\mathbb{R}^n$-coordinate components of the projection $\iota(s) = \rho(s) \cdot s: S \to \mathcal{K}$ are smooth invariant functions, called normalized invariants. Normalized invariants contain a maximal set of functionally independent invariants, and have a *replacement property*, which allows to rewrite any invariant in terms of them by simple substitution.[17, 26, 27]

Although, a global smooth cross-section does not always exist, a local smooth cross-section[§] passing through every point of S may be found for every semi-regular action.[¶] The freeness assumption can be also relaxed to

[‡]Although plotting invariants as a function of the affine arc-length eliminates the dependence on a parameter, it is more difficult in practice, and the dependence on a choice of the initial point remains.

[§]A local cross-section is defined on an open subset of $U \subset S$ and $\forall s \in U$ intersects each connected component of $O_s \cap U$ at a unique point.

[¶]An action of G is called *semi-regular* if all orbits have the same dimension.

a semi-regularity assumption. With these weaker assumptions the above method can be used to construct local invariants.[17, 27‖]

For algebraic groups acting on algebraic varieties, a purely algebraic counterpart of the Fels-Olver construction was obtained in Ref. 26, 27. This formulation gives rise to a computer-algebra algorithm for constructing a generating set of rational invariants and for constructing replacement invariants. The latter are algebraic over the field of rational invariants, and play the same role as normalized invariants in the smooth construction. The advantage of the algebraic construction is that it avoids a generically non-constructible step of computing a moving frame map ρ. The algorithms rely on a Gröbner basis computation. The algebraic method can be combined with the inductive approach described below. In some particular examples, including the 3D example presented here, the computation based on the moving frame map ρ turns out to be more practical.

When the group G is of relatively large dimension, computation of invariants by either a geometric or algebraic approach becomes challenging. In Ref. 16 two modifications of the above method were proposed. These allow to split the computation into two steps: first invariants of a subgroup $A \subset G$ are computed, and then invariants of the entire group are constructed in terms of those. For the problem at hand, we use one of these modifications, called the inductive approach, which is applicable when a group factors into a product of two subgroups.

4.2. Inductive approach

Definition 4.1. A group G *factors as a product* of its subgroups A and B if for any $g \in G$ there are $a \in A$ and $b \in B$ such that $g = ab$.

We write $G = A \cdot B$. If in addition $A \cap B = e$, then for each $g \in G$ there are *unique* elements $a \in A$ and $b \in B$ such that $g = ab$.

Provided there is a cross-section $\mathcal{K}_A$, containing s, invariant under the action of the subgroup B, it follows from Lemma 4.7 in Ref. 16 that invariants of G can be constructed from the invariants of A using the following method.

Inductive method:

1. Restrict the G-action to A. Find a local cross-section $\mathcal{K}_A \subset S$ for the action of A which is invariant under the action of B.

2. Construct a moving frame map $\rho_A\colon S \to A$ defined by the condition $\rho_A(s) \cdot s \in \mathcal{K}_A$, $\forall s \in S$, by solving the corresponding equations. Composition of coordinate functions with the projection $\iota(s) = \rho_A(s) \cdot s\colon S \to \mathcal{K}_A$ are invariant with respect to the action of A.

3. Restrict the action of G to the action of its subgroup B on the invariant subset $\mathcal{K}_A$ and choose a local cross-section $\mathcal{K}_B \subset \mathcal{K}_A$.

4. Construct a moving frame map $\rho_B\colon \mathcal{K}_A \to B$ defined by the condition $\rho_B(s) \cdot z \in \mathcal{K}_B$, $\forall z \in \mathcal{K}_A$, by solving the corresponding equations.

5. The G-moving frame map $\rho\colon S \to G$ is defined by $\rho(s) = \rho_B(\rho_A(s) \cdot s)\rho_A$, and G-invariants coordinate components of $\rho(s) \cdot s = \rho_B(\rho_A(s) \cdot s) \cdot (\rho_A(s) \cdot s) = \rho_B(\iota_A(s)) \cdot \iota_A(s)$.

Example 4.2. INTEGRAL AFFINE INVARIANTS FOR CURVES IN 2D.

To illustrate the inductive approach we start with a simpler 2D case. The standard action of $SL(2) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} | a_{22}a_{11} - a_{21}a_{12} = 1 \right\}$ on $\mathbb{R}^2$ prolongs to the integral variables of order 2. It is sufficient to consider three integral variables $Y_{10} = \int_0^t X dY$, $\quad Y_{11} = \int_0^t XY dY$, $\quad X_{11} = \int_0^t XY dX$. The other 3 second order

‖A function f, defined on an open subset U of S, is a *local invariant* if $\forall s \in U$ there exists an open neighborhood G_s of $e \in G$ s.t. condition (2) is satisfied for all $g \in G_s$.

integral variables are related to those via integration by parts and therefore need not be considered. We obtain the following free action on an open subset of $M = \mathbb{R}$.

$$
\begin{aligned}
\overline{X} &= a_{11}X + a_{12}Y \\
\overline{Y} &= a_{21}X + a_{22}Y \\
\overline{Y_{10}} &= \int_0^t \overline{X}d\overline{Y} = Y_{10} + a_{11}a_{21}\frac{X^2}{2} + a_{12}a_{22}\frac{Y^2}{2} + a_{12}a_{21}XY \\
\overline{Y_{11}} &= \int_0^t \overline{XY}d\overline{Y} \\
&= \frac{{a_{21}}^2 a_{11}X^3}{3} - a_{21}X_{11} + a_{21}a_{12}a_{22}XY^2 + a_{22}a_{11}a_{21}X^2Y + a_{22}Y_{11} + \frac{{a_{22}}^2 a_{12}Y^3}{3} \\
\overline{X_{11}} &= \int_0^t \overline{XY}d\overline{X} \\
&= \frac{{a_{11}}^2 a_{21}X^3}{3} + a_{11}X_{11} + a_{11}a_{12}a_{22}XY^2 - a_{12}Y_{11} + a_{12}a_{11}a_{21}X^2Y + \frac{{a_{12}}^2 a_{22}Y^3}{3}
\end{aligned}
\tag{10}
$$

We have a product decomposition $SL(2) = B \cdot A$, where $B = \left\{ \begin{pmatrix} b_{11} & b_{12} \\ 0 & \frac{1}{b_{11}} \end{pmatrix} | b_{11} > 0 \right\}$ and $A = SO(2)$ is a group of rotations. The intersection $B \cap A = \{e\}$, and therefore we can apply the inductive method as follows.

1. We restrict the action (10) to A:

$$
\begin{aligned}
\overline{X} &= \cos\phi\, X - \sin\phi\, Y \\
\overline{Y} &= \sin\phi\, X + \cos\phi\, Y \\
\overline{Y_{10}} &= Y_{10} + \frac{1}{2}\cos\phi\sin\phi\, \left(X^2 - Y^2\right) - \sin^2\phi XY \\
\overline{Y_{11}} &= \cos\phi Y_{11} - \sin\phi X_{11} \\
&+ \frac{1}{3}\cos\phi\sin\phi\, \left(\sin\phi X^3 + 3\,\cos\phi X^2Y - 3\,\sin\phi XY^2 - \cos\phi Y^3\right) \\
\overline{X_{11}} &= \cos\phi X_{11} + \sin\phi Y_{11} \\
&+ \frac{1}{3}\cos\phi\sin\phi\, \left(\cos\phi X^3 - 3\,\sin\phi X^2Y - 3\,\cos\phi XY^2 + \sin\phi Y^3\right)
\end{aligned}
\tag{11}
$$

   A subset $\mathcal{K}_A$ defined by conditions, $Y = 0, X > 0$ serves as a cross-section on the subset of $\mathbb{R}^5$, where $X^2 + Y^2 \neq 0$. Moreover $\mathcal{K}_A$ is invariant under the restriction of (10) to subgroup B.

2. The corresponding moving frame map $\rho_A(s) = \begin{pmatrix} \frac{X}{\sqrt{X^2+Y^2}} & \frac{Y}{\sqrt{X^2+Y^2}} \\ -\frac{Y}{\sqrt{X^2+Y^2}} & \frac{X}{\sqrt{X^2+Y^2}} \end{pmatrix}$ is obtained by solving the equation $\overline{Y} = 0$ with the condition $\overline{X} > 0$. The projection $\iota_A\colon \mathbb{R}^5 \to \mathcal{K}_A$, obtained by substitution ρ_A into (11) produces a point whose coordinates are invariant under the action of $SO(2)$:

$$
\begin{array}{lll}
X_A = \sqrt{X^2+Y^2}, & Y_A = 0, & Y_{10\,A} = Y_{10} - \frac{XY}{2}, \\
Y_{11\,A} = -\frac{2\,Y^2X^2 - 3\,XY_{11} - 3\,YX_{11}}{3\sqrt{X^2+Y^2}}, & & X_{11\,A} = -\frac{YX^3 - 3\,XX_{11} - XY^3 + 3\,YY_{11}}{3\sqrt{X^2+Y^2}}.
\end{array}
\tag{12}
$$

3. We now restrict the action (10) to the action of a subgroup B on an invariant subset $\mathcal{K}_A$. We obtain the following transformations:**

$$
\begin{array}{ll}
\overline{X_A} = b_{11}X_A, & \overline{Y_{10\,A}} = Y_{10\,A}, \\
\overline{Y_{11\,A}} = \frac{1}{b_{11}}Y_{11\,A}, & \overline{X_{11\,A}} = b_{11}X_{11\,A} - b_{12}Y_{11\,A}.
\end{array}
\tag{13}
$$

** At this step of the construction we treat $X_A, Y_{10\,A}, Y_{11\,A}, X_{11\,A}$ as coordinate functions on $\mathcal{K}_A$ disregarding their expressions (12) in terms of $X, Y, Y_{10}, Y_{11}, X_{11}$.

A subset $\mathcal{K}_B \subset \mathcal{K}_A$ defined by the equations $X_A = 1, \quad X_{11\,A} = 0$ serves as a cross-section on the subset of $\mathcal{K}_A$, where $Y_{11\,A} \neq 0$.

4. This leads to the moving frame map $\rho_B(s) = \begin{pmatrix} \frac{1}{X_A} & \frac{X_{11\,A}}{Y_{11\,A}X_A} \\ 0 & X_A \end{pmatrix}$. The projection $\iota_B : \mathcal{K}_A \to \mathcal{K}_B$, defined by $\iota_B(s) = \rho_B(s) \cdot s$, produces a point with coordinates

$$X_B = 1, \quad Y_{10\,B} = Y_{10\,A}, \quad Y_{11\,B} = X_A Y_{11\,A}, \quad X_{11\,B} = 0$$

invariant under the B-action (13) on $\mathcal{K}_A$.

5. Substitution of (12) produces 2 non-constant independent invariants under the $SL(2)$ action (10).

$$I_1 = Y_{10} - \frac{XY}{2}, \quad I_2 = X\,Y_{11} + Y\,X_{11} - \frac{2}{3}\,Y^2X^2 \tag{14}$$

By considering the effect of scaling on this invariants, we obtain a single $GL(2)$ invariant:

$$I = \frac{3}{4}\,\frac{I_2}{I_1^2} = \frac{2\,Y^2X^2 - 3\,XY_{11} - 3\,YX_{11}}{(2Y_{10} - XY)^2}$$

By substituting (7), one obtains an invariant with respect to action of *full affine group*, equivalent to the one obtained in.[13]

4.3. Affine integral invariants for curves in 3D.

We start by considering the action of $SL(3)$ defined by (6) on the integral variables of order 2. Taking into account the relations which arise from the integration by parts, it is sufficient to consider the following 11 integral variables: Z_{100}, Z_{010}, Y_{100}, Z_{011}, Z_{020}, Z_{101}, Z_{110}, $Y_{101}\,X_{110}$, $X_{101}\,X_{020}$. We therefore obtain a free action of $SL(3)$ on an open subset of $\mathbb{R}^{14}$. We have a product decomposition $SL(3) = B \cdot A$, where $B = \left\{ \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & \frac{1}{b_{11}b_{22}} \end{pmatrix} | b_{11} > 0 \right\}$ and $A = SO(3)$ is a group of rotations. The intersection $B \cap A = \{e\}$ is trivial. We again follow the steps of the inductive method.

1. We restrict the action (6) to A whose elements can be represented as the product of three rotations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\psi) & -sin\psi \\ 0 & sin\psi & cos\psi \end{pmatrix} \begin{pmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -sin\phi & 0 & cos\phi \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A subset $\mathcal{K}_A$, defined by conditions, $Y = 0$, $Z = 0$, $Z_{011} = 0, X > 0$ serves as a cross-section on the subset of $\mathbb{R}^{14}$, where $X^2 + Y^2 + Z^2 > 0$. The cross-section $\mathcal{K}_A$ is invariant under the action of B.

2. The corresponding moving frame map ρ_A is obtained by solving the equation $\overline{Y} = 0, \overline{Z} = 0, \overline{Z_{011}} = 0$ with the condition $\overline{X} > 0$. Explicitly

$$\begin{array}{lll} \cos\theta = \frac{X}{\sqrt{X^2+Y^2}}, & \cos\phi = \frac{\sqrt{X^2+Y^2}}{\sqrt{X^2+Y^2+Z^2}}, & \cos\psi = \frac{Z_{020_R}}{\sqrt{{Z_{020_R}}^2 + 4\,{Z_{011_R}}^2}}, \\ \sin\theta = -\frac{Y}{\sqrt{X^2+Y^2}}, & \sin\phi = \frac{Z}{\sqrt{X^2+Y^2+Z^2}}, & \sin\psi = -2\,\frac{Z_{011_R}}{\sqrt{{Z_{020_R}}^2 + 4\,{Z_{011_R}}^2}}, \end{array} \tag{15}$$

where

$$\begin{aligned} Z_{011_R} \quad = \quad & -\frac{1}{6\sqrt{X^2+Y^2}\,(X^2+Y^2+Z^2)} \\ & (-6\,X^3Z_{011} + 2\,X^3YZ^2 + 6\,X^2YZ_{101} - 6\,ZX^2Y_{101} - 6\,XZ_{011}Y^2 + 4\,Y^3Z^2X \\ & +6\,ZXYX_{101} - 6\,XZ^2X_{110} + 3\,XZYZ_{020} - 6\,ZY^2Y_{101} - 3\,Z^2YX_{020} + 6\,Y^3Z_{101} - 6\,Y^2ZZ_{110}) \end{aligned}$$

and

$$Z_{020_R} = -\frac{1}{3}\,\frac{-2\,Y^2ZX^2 - 3\,X^2Z_{020} + 3\,XZX_{020} + 6\,YXZ_{110} - 6\,ZYX_{110} + 6\,Y^2X_{101}}{\sqrt{X^2+Y^2}\sqrt{X^2+Y^2+Z^2}}$$

The coordinate components of the projection $\iota_A\colon \mathbb{R}^{14} \to \mathcal{K}_A$, obtained by substitution of $\rho_A(s)$ into (6),

$$X_A = \sqrt{X^2+Y^2+Z^2},\ Y_A = 0,\ Z_A = 0,\ Z_{010A} = \frac{XYZ - 2\,XZ_{010} + 2\,YZ_{100} - 2\,ZY_{100}}{2\sqrt{X^2+Y^2+Z^2}}, \tag{16}$$
$$Z_{100A} = \ldots,\ Y_{100A} = \ldots,\ Z_{011A} = \ldots,\ Z_{020A} = \ldots,\ Z_{101A} = \ldots,\ Z_{110A} = \ldots,\ Y_{101A} = \ldots$$

are invariant under $SO(3)$ action.[††]

3. We now restrict the action (6) to the action of a subgroup B on an invariant subset $\mathcal{K}_A$. We obtain the following transformations:

$$\begin{aligned}
\overline{X_A} &= b_{11}\,X_A\\
\overline{Z_{010A}} &= b_{22}\,b_{33}\,Z_{010A}\\
\overline{Z_{100A}} &= b_{11}\,b_{33}\,Z_{100A} + b_{12}\,b_{33}\,Z_{010A}\\
\overline{Y_{100A}} &= b_{11}\,b_{22}\,Y_{100A} - b_{13}\,b_{22}\,Z_{010A} + b_{11}\,b_{23}\,Z_{100A} + b_{12}\,b_{23}\,Z_{010A}\\
\overline{Z_{020A}} &= {b_{22}}^2 b_{33}\,Z_{020A}\\
\overline{Z_{101A}} &= b_{11}\,{b_{33}}^2 Z_{101A}\\
\overline{Z_{110A}} &= b_{33}\,b_{11}\,b_{22}\,Z_{110A} + b_{33}\,b_{11}\,b_{23}\,Z_{101A} + b_{33}\,b_{12}\,b_{22}\,Z_{020A}\\
\overline{Y_{101A}} &= Y_{101A} + \frac{b_{23}}{b_{22}} Z_{101A} - \frac{b_{12}}{2b_{11}} Z_{020A}
\end{aligned}$$

A subset $\mathcal{K}_B \subset \mathcal{K}_A$ defined by equations $Z_{010A} = 1$, $Z_{100A} = 1$, $Y_{100A} = 1$, $Z_{020A} = 1, Z_{110A} = 1$ is a cross-section on a subset of $\mathcal{K}_A$, where Z_{020A}, Z_{010A} and Z_{101A} are non-zero.

4. The corresponding moving frame map ρ_B is obtained by solving equations $\overline{Z_{010A}} = 1$, $\overline{Z_{100A}} = 1$, $\overline{Y_{100A}} = 1$, $\overline{Z_{020A}} = 1, \overline{Z_{110A}} = 1$:

$$b_{11} = Z_{010A}, b_{12} = -\frac{Z_{020A}Z_{100A} - Z_{010A}}{Z_{020A}}, b_{22} = \frac{Z_{010A}}{Z_{020A}}, b_{23} = \frac{-Z_{010A}Z_{110A} + Z_{020A}Z_{100A}}{Z_{020A}Z_{101A}}$$
$$b_{13} = -\frac{-{Z_{010A}}^2 Z_{020A}Z_{100A} + {Z_{010A}}^3 Z_{110A} - Z_{020A}Z_{101A}{Z_{010A}}^2 Y_{100A} + Z_{101A}{Z_{020A}}^2}{Z_{020A}Z_{101A}{Z_{010A}}^2}$$

The coordinate components of the projection $\rho_B(s)\cdot s\colon \mathcal{K}_A \to \mathcal{K}_B$

$$X_B = Z_{010A}X_A, \quad Z_{101B} = \frac{Z_{101A}Z_{020A}^2}{Z_{010A}^3}, \quad Y_{101B} = \frac{2Y_{101A}Z_{010A} - 2Z_{010A}Z_{110A} + 3Z_{020A}Z_{100A}}{2Z_{010A}} - \frac{1}{2}$$

are invariant under the action of B on $\mathcal{K}_A$.

5. Substitution of explicit expressions (16) produces 3 non-constant invariants under the action of $SL(2)$. The expression for the invariant corresponding to Z_{101_B} is complicated and is not presented in this paper. The invariants corresponding to X_B and $Y_{101B} + \frac{1}{2}$ are

$$\begin{aligned}
i_1 &= \frac{n_1X + n_2Z - n_3Y}{4}\\
i_2 &= \frac{-1}{4i_1}(2\,n_1(XYZ^2 - 3\,Z_{011}X + 3\,YZ_{101} - ZZ_{110} - 2\,ZY_{101}) + n_2(2\,XY^2Z + 3\,XZ_{020}\\
&- 6\,ZX_{020} - 4\,YZ_{110} - 2\,YY_{101}) - 2\,n_3(3\,YX_{101} - 3\,ZX_{110} + XZ_{110} - XY_{101}))
\end{aligned}$$

[††]Due to space limitation we omit some of the explicit formulae, which unlike the final result are complicated.

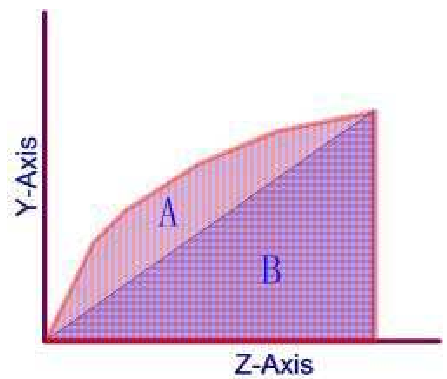


Figure 4. Geometric interpretation of term n_1

where

$$n_1 = YZ - 2\,Z_{010}, \quad n_2 = XY - 2\,Y_{100}, \text{ and } n_3 = XZ - 2\,Z_{100}$$

represent certain areas. In particular, if a curve on Fig. 4 represents the projection of a given space curve to ZY-plane, then the term n_1 is two times the difference of the area under the curve and the area of the triangle B.

The quotient $I = \frac{i_2}{i_1}$ is invariant under $GL(3)$. We used $I_1 = 4i_1$ and $I_2 = -4i_1 i_2$ to construct signatures in Section 3.

5. CONCLUSION

Using the inductive approach to the moving frame construction, we derived integral invariants for 3D curves transformed by the special and the full affine groups. These invariants are used to construct signatures that classify curves up to the special affine transformations. Further research includes a detailed study of the properties of integral signatures for curves in 2D and 3D. From a theoretical perspective this involves proving the separation properties of integral signatures. We will also investigate the symmetry detection using such signatures. Bearing in mind applications to computer vision and image processing, we will present a quantitative analysis of the sensitivity of the integral invariants to noise. Derivation of the integral invariants with respect to the projective group, as well as the extension of the above methods to surfaces, is a natural but computationally challenging continuation of the research.

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