TWO ALGORITHMS FOR A MOVING FRAME CONSTRUCTION.

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Abstract. The method of moving frames, introduced by Elie Cartan, is a powerful tool for the solution of various equivalence problems. The practical implementation of Cartan's method, however, remains to be challenging, despite its later significant development and generalization. This paper presents two variations on the Fels and Olver algorithm, which under some conditions on the group action, simplify a moving frame construction. In addition, the first algorithm leads to a better understanding of invariant differential forms on the jet bundles, while the second expresses the differential invariants for the entire group in terms of the differential invariants of its subgroup.

1. Introduction

Elie Cartan's method of equivalence [5] is a natural development of the Felix Klein Erlangen program (1872), which describes geometry as the study of invariants of group actions on geometric objects. Classically, a moving frame is an equivariant map from the space of submanifolds (or more rigorously, from the corresponding jet bundle) to the bundle of frames. Exterior differentiation of this map produces a number of differential invariants. Differential invariants provide a key to the solution of many equivalence problems and are also used in the process of reduction of differential equations and variational problems (see for instance [6], [12], [17], [19], [2] and [18]).

Considering moving frame constructions on homogeneous spaces, Griffiths [15] and Green [14] observed that a moving frame can be viewed as an equivariant map from the space of submanifolds to the group itself. Adopting this observation as a general definition of a moving frame, Fels and Olver [10], [11] generalized the Cartan's method to arbitrary, not necessarily transitive, finite-dimensional Lie group actions on a manifold, introducing a simple algorithm for constructing moving frames and differential invariants. Following this algorithm, one prolongs the action to a jet space of sufficiently high order to obtain a system of algebraic equations on the group parameters, whose solutions lead to a moving frame. This last step might become trivial or very difficult depending on the group action we consider. In the earlier methods [15] and [14], however, the moving frame is constructed gradually – at each order of prolongation one normalizes some of the group parameters at the end obtaining a moving frame for the entire group. We combine the advantages of both approaches in the recursive algorithm presented here. This approach, along with the known result [23] cited in Proposition 3.8, leads to a description of the structure of invariant differential forms on the jet bundles, the question raised in [9].

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Not surprisingly, the construction of moving frames and differential invariants is simpler when the acting group has fewer parameters. Thus, it is desirable to use the results obtained for a subgroup $A \subset G$ to construct a moving frame and differential invariants for the entire group G. The inductive algorithm presented here allows us, in the case when the group G factors as a product, to extend a moving frame for a subgroup to the entire group. As a byproduct one obtains at the same time the relations among the invariants of group G and its subgroup A. It worth remarking that, in order to obtain such relations, the algorithm does not require the explicit formulae for the invariants of either G or A, but only the corresponding moving frames (or normalizations) which lead to these invariants. We illustrate the algorithm by making an induction from the Euclidean action on plane curves to the special affine action, and then to the action of the entire projective group, which leads to the expression of the affine invariants in terms of the Euclidean ones and the projective invariants in terms of the affine ones. These are classical actions whose differential invariants are well known (for instance, see [3], [7], [10]). The actions of all three groups play an important role in computer image processing [8], [22]. We also include the derivation of the affine invariants in terms of the Euclidean for curves in $\mathbb{R}^3$.

2. The Method of Moving Frames

Given a manifold M of dimension m and an integer $1 \leq p \leq m$, we let $J^k = J^k(M,p)$ denote the k-th order jet bundle, whose fiber over $z \in M$ consists of equivalence classes of p-dimensional submanifolds of M under the equivalence relation of k-th order contact at z. The infinite jet bundle $J^\infty = J^\infty(M,p)$ is defined as the inverse limit of the finite jet bundles under the standard projections $\pi^{k+1}_k \colon J^{k+1} \to J^k$. We will identify functions and differential forms on J^k with their pull backs to any higher order jets including J^∞.

Let U be a coordinate chart on M. We arbitrarily divide the set of coordinate functions on U into two subsets: the set of independent variables $x^1, ..., x^p$ and the set of dependent variables $u^1, \dots u^q$, where $p + q = m$. The k-th jets of all submanifolds $S \subset U$ which satisfy the transversality condition $dx^1 \wedge \cdots \wedge dx^p|_S \neq 0$ form a coordinate chart $U^k \subset J^k$ which can be parameterized by coordinate functions $x^1, \dots, x^p$, u^α_J, where $i = 1, \dots, p$, $\alpha = 1, \dots, q$ and $J = (j_1, \dots, j_k)$, with $0 \leq j_\nu \leq p$, is a symmetric multi-index of length $|J| = k$.

The cotangent bundle over J^∞ has a distinguished sub-bundle $\mathcal{C}$, whose sections are identically zero when restricted to a jet of any p-dimensional submanifold of M. In local coordinates $\mathcal{C}$ is spanned by the forms $\theta^\alpha_J = du^\alpha_J - \sum_i^p u^\alpha_{J,i} dx^i$, $\alpha = 1, \dots, q$, $0 \leq |J|$. The differential ideal generated by one-forms in $\mathcal{C}$ is called *contact ideal*. On a local chart we can define a complementary horizontal sub-bundle H spanned by the forms $dx^1, \dots, dx^p$. This splitting induces a bigrading on the algebra of differential forms $\bigwedge T^* J^\infty$. For any differential form λ, we let $\pi_H \lambda$ denote its purely horizontal component and $\pi_V \lambda$ denote its purely contact component. There is also a corresponding splitting of the tangent bundle over J^∞. In particular, the vector fields on J^∞, which are annihilated by any contact form, form a sub-bundle of *total* (or horizontal) vector fields.

A smooth action of a Lie group G on M can be uniquely prolonged to a smooth action on J^∞ under the condition that it preserves contact ideal. By definition, a

k-th order *differential invariant* of G is a function on J^k which is invariant under the prolonged action.

We will review the basic steps of the moving frame construction presented in [11] by Fels and Olver.

Definition 2.1. A k-th order (right) moving frame is a smooth right G-equivariant map $\rho^{(k)}$ from an open subset of J^k to G:

$$\rho^{(k)}(g \cdot z^{(k)}) = \rho^{(k)}(z^{(k)}) \cdot g^{-1}, \tag{1}$$

for all $g \in G$ such that $z^{(k)}$ and $g \cdot z^{(k)} \in J^k$ are in the domain of definition of ρ.

The existence of a moving frame on a jet bundle can be deduced from the following two theorems. See [11] for a proof of the first theorem and [21], [20], for a proof of the second one.

Theorem 2.2. *Let a Lie group G act on a manifold N. Then there exists a smooth G-equivariant map from a neighborhood of each point in N to the group G if and only if G acts freely and regularly.*

The regularity condition in the above theorem means that every point of N has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

Theorem 2.3. *Let G be a Lie group that acts locally effectively on each open subset of M. Then there is a minimal order $n \leq r = \dim G$, such that the prolonged action of G on J^k is locally free on some open and dense subset $\mathcal{V}^k \subset J^k$ for each $k \geq n$.*

By definition, *local freeness* of the action means that the isotropy group of each point is discrete. The order n in the theorem above is called the *order of stabilization* and the subsets $\mathcal{V}^k$ are called *regular*.

We notice that the conclusion of the second theorem is weaker than the assumption of the first one. It guarantees, however, that all orbits on $\mathcal{V}^n$ have the same dimension $r = \dim G$. Using the Frobenius' theorem, one can construct a submanifold $\mathcal{K}^n$, which is transversal the orbits on an open neighborhood of a point $z^{(n)} \in \mathcal{V}^n$, and has complementary dimension. Such manifold is called a *cross-section* to the orbits. (See the proof of Theorem 4.4 for a similar construction.) If the action is regular then by shrinking $\mathcal{K}^n$ we can make it intersect each orbit no more than once. Let us assume for a moment that the action is free and regular. Then the moving frames (1) near $\mathcal{K}^n$ is defined by the condition

$$\rho(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}^n$$

Since each orbit intersects $\mathcal{K}^n$ at a unique point, then

$$\rho(z^{(n)}) \cdot z^{(n)} = \rho(g \cdot z^{(n)}) \cdot (g \cdot z^{(n)}) \tag{2}$$

and this leads to the right equivariance condition (1) due to the freeness assumption.

The cross-section $\mathcal{K}^n$ and the moving frame ρ can be extended to any higher order regular set $\mathcal{V}^k$, including $\mathcal{V}^\infty \subset J^\infty$, by defining $\mathcal{K}^k = \{z^{(k)} | \pi^k_n z^{(k)} \in \mathcal{K}^n\}$ and $\rho(z^{(k)}) = \rho\left(\pi^k_n(z^{(k)})\right)$ for $k = n, \ldots, \infty$.

Remark 2.4. For a locally free, not necessarily regular action on $\mathcal{V}^n$, which is guaranteed by Theorem (2.3), a moving frame can be defined in a similar fashion. In this case, however, the equivariant condition (1) will hold only when g belongs to some open neighborhood of the identity in G which may depend on $z^{(n)}$.

Remark 2.5. *Despite the locality of the moving frame definition, we will adopt a global notation, therefore writing* $\rho\colon J^\infty \to G$*, while, in fact, the domain and the range of* ρ *are some open subsets of* J^∞ *and* G *respectively.*

Moreover, the differential invariants and invariant differential forms which appear further might be defined only locally, on an open subset of J^∞*, and be invariant only with respect to group elements in some open neighborhood of the identity. Such invariants are often referred as* infinitesimal invariants*, since they can be defined by the condition that their Lie derivatives with respect to any infinitesimal generator of the action vanish.*

Given a moving frame, one can define a process of invariantization (see [11], [18]) which will project the space of differential forms (in particular functions) on J^∞ onto the space of invariant differential forms (functions). We start by lifting the prolonged G action to the space $\mathcal{B} = G \times J^\infty$:

$$h \cdot (g, z^{(\infty)}) = (gh^{-1}, h \cdot z^{(\infty)}),$$

where g, $h \in G$. We also introduce maps $w\colon \mathcal{B} \to J^\infty$ to be defined by the prolonged group action: $w(g, z^{(\infty)}) = g \cdot z^{(\infty)}$, and $\sigma\colon J^\infty \to \mathcal{B}$ to be defined via a moving frame: $\sigma(z^{(\infty)}) = (\rho(z^{(\infty)}), z^{(\infty)})$. We note that w is a G-invariant map, while σ is a G-equivariant map. Thus their composition $w \circ \sigma(z^{(\infty)}) = \rho(z^{(\infty)}) \cdot z^{(\infty)}$ is a G-invariant projection $J^\infty \longrightarrow \mathcal{K}^\infty$.

The cotangent bundle $T^*\mathcal{B}$ over $\mathcal{B}$ is a direct sum of the bundles T^*G and T^*J^∞. This induces a bigrading on $\bigwedge T^*\mathcal{B}$. For a differential form $\tilde\lambda$ on $\mathcal{B}$ we let $\pi_G\tilde\lambda$ denote the purely group component of $\tilde\lambda$ and $\pi_J\tilde\lambda$ its purely jet component. If $\tilde\lambda$ is a one-form then $\tilde\lambda = \pi_G\tilde\lambda + \pi_J\tilde\lambda = \pi_G\tilde\lambda + \pi_H\tilde\lambda + \pi_V\tilde\lambda$.

Definition 2.6. The invariantization of a differential form λ on J^∞ is the invariant differential form

$$\iota(\lambda) = \sigma^*\left(\pi_J(w^*\lambda)\right). \tag{3}$$

In the case of functions (zero forms) (3) reduces to

$$\iota(f)(z^{(\infty)}) = \sigma^* w^*(f)(z^{(\infty)}) = f\left(\rho(z^{(\infty)}) \cdot z^{(\infty)}\right). \tag{4}$$

Geometrically, invariantization of a differential form λ (or function f) is the unique invariant differential form (function) which agrees with λ (or f) on the cross-section $\mathcal{K}^\infty$. We note also that both $w^*\lambda$ and $\pi_J w^*\lambda$ are invariant forms on $\mathcal{B}$.

Invariantization of the coordinate functions:

$$H^i = \iota(x^i),\ i = 1, \ldots, p, \qquad I^\alpha_J = \iota(u^\alpha_J),\ \alpha = 1, \ldots, q,$$

provide a complete (or fundamental) set of local differential invariants on J^∞, in a sense that every other local differential invariant can be expressed as a function of these invariants.

Invariantization of the basis one-forms $dx^1, ..., dx^p$, θ^α_J:

$$\begin{aligned} \varpi^i &= \iota(dx^i) = \sigma^*\, d_J\, w^*\,(x^i),\ \ i = 1, \ldots, p, \\ \vartheta^\alpha_J &= \iota(\theta^\alpha_J) = \sigma^*\, \pi_J\, w^*\,(\theta^\alpha_J),\ \ \alpha = 1, \ldots, q \end{aligned}$$

produces an invariant coframe on J^∞. We note that invariantization preserves the contact sub-bundle $\mathcal{C}$ of T^*J^∞, but the horizontal sub-bundle H is not generally preserved under invariantization. We can decompose $\varpi^i = \iota(dx^i) = \omega^i + \eta^i$, $i = 1, \dots, p$, where the non-zero horizontal forms

$$\omega^i = \sigma^* \, \pi_H \, w^*(dx^i) = \sigma^* \, d_H \, w^* \, (x^i) \tag{5}$$

are invariant up to a contact form, that is, $g^*\omega^i = \omega^i + \Theta^i$, for some contact one-forms Θ^i. Forms with such transformation property are called *contact invariant.* By adding contact forms η^i to ω^i one obtains fully invariant forms ϖ^i. Forms $\omega^i, i = 1, \dots, p$ are linearly independent. The total vector fields $\mathcal{D}_i, i = 1, \dots, p$, dual to ω^i, form a complete set of *invariant differential operators*, which map differential invariants to differential invariants of higher order. See [19] for further details.

Example 2.7. Let us consider the action of the special Euclidean group $SE(2) = SO(2) \ltimes R^2$ on plane curves $u = u(x)$. Its first prolongation, given by

$$\begin{aligned} x &\mapsto \cos(\phi)x - \sin(\phi)u + a, \\ u &\mapsto \sin(\phi)x + \cos(\phi)u + b, \\ u_x &\mapsto \frac{\sin(\phi) + \cos(\phi)u_x}{\cos(\phi) - \sin(\phi)u_x} \end{aligned} \tag{6}$$

defines a free action on $J^1(\mathbb{R}^2, 1)$. A moving frame on $J^1(\mathbb{R}^2, 1)$ can be obtained by choosing a cross-section $\{x = 0,\, u = 0,\, u_x = 0\}$. Then an equivariant map $J^1(\mathbb{R}^2, 1) \to SE(2)$ is found by setting expressions (6) equal to zero and solving for the groups parameters:

$$\phi = -\arctan(u_x), \qquad a = -\frac{u_x u + x}{\sqrt{1 + u_x^2}}, \qquad b = \frac{u_x x - u}{\sqrt{1 + u_x^2}}. \tag{7}$$

The corresponding element of the special Euclidean group can be written in a matrix form:

$$\rho = \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{u u_x + x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{x u_x - u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

A fundamental set of k-th order differential invariants can be obtained by prolonging the action to J^k and normalizing the group parameters, that is by substituting (7) into the formulae. For instance, the forth order prolongation is given by:

$$\begin{aligned} u_{xx} &\mapsto \frac{u_{xx}}{\Delta^3}, \\ u_{xxx} &\mapsto \frac{\Delta u_{xxx} + 3\sin(\phi) u_{xx}^2}{\Delta^5}, \\ u_{xxxx} &\mapsto \frac{\Delta^2 u_{xxxx} + 10 \sin(\phi) \Delta u_{xx} u_{xxx} + 15 \sin^2(\phi) u_{xx}^3}{\Delta^7}, \end{aligned} \tag{8}$$

where $\Delta = \cos(\phi) - \sin(\phi) u_x$. Substitution of (7) into (8) produces fourth order differential invariants:

$$\begin{aligned} I_2^e &= \frac{u_{xx}}{(1 + u_x^2)^{3/2}}, \\ I_3^e &= \frac{(1 + u_x^2) u_{xxx} - 3 u_x u_{xx}^2}{(1 + u_x^2)^3}, \\ I_4^e &= \frac{(1 + u_x^2)^2 u_{xxxx} - 10 u_x u_{xx} u_{xxx} (1 + u_x^2) + 15 u_x^2 u_{xx}^3}{(1 + u_x^2)^{9/2}}. \end{aligned} \tag{9}$$

We note that $I_2^e = \kappa$, the Euclidean curvature, $I_3^e = \kappa_s = \frac{d\kappa}{ds}$, where $ds = \sqrt{1+u_x^2}\,dx$ is infinitesimal arc length, but $I_4^e = \kappa_{ss} + 3\kappa^3$ (instead of just κ_{ss}), according to recurrence formula (13.4) in [11]. The contact invariant differential form equals to $\omega = \sigma^*(d_H\, w^* x) = \sqrt{1+u_x^2}\,dx = ds$. The dual total vector field $\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}} D_x = \frac{d}{ds}$ provide an invariant differential operator, such that any other invariant can be expressed as a function of κ and its derivatives with respect to $\mathcal{D}$.

Remark 2.8. Lack of space precludes us from a detailed comparison of the classical method of moving frames as presented, for instance, in [5] and [16], with its generalization [11] described above. We note, however, that all classical moving frames lead to equivariant maps from a jet bundle to the group under consideration, while certainly not every such map can be described as an invariant section of a frame bundle. Classical differential invariants are obtained by pulling back the invariant coframe on G under this equivariant map, which may lead to a different (but equivalent) set of fundamental invariants.

3. Recursive Construction of Moving Frames.

Assume that a smooth action of G on M is regular but not free. Using Frobenius' theorem, one can still find a local cross-section $\mathcal{K}$ to the orbits of G and define a local map $\rho_0 \colon M \to G$ by the condition $\rho_0(z) \cdot z \in \mathcal{K}$ for $z \in M$. Then the non-constant coordinates of $\rho_0(z) \cdot z$ provide a complete set of zero order differentials invariants. We remind the reader that despite our global notation, the whole construction is in fact local (see Remark 2.5). At the next stage we would like to obtain invariants of strictly higher order and also to use the map ρ_0 as a building block for the moving frame. The recursive algorithm presented here allows us to do so, provided we require each point of $\mathcal{K}$ have the same isotropy group. The obvious necessary condition for the existence of such cross-section in a neighborhood U of $z_0 \in M$ is that the isotropy groups of any two points in U are conjugate by an element of G, or in other words, all orbits are of *the same type*. There is a simple counterexample of this phenomenon ([24], Example 1,§ 7).

Example 3.1. Let $\mathbb{R}^2$ act on $\mathbb{R}^2$ by

$$x \to x + au + b, \qquad u \to u$$

The isotropy group of a point (x_0, u_0) is defined by the condition $au_0 + b = 0$. The orbits are lines parallel to the x-axis. All the points that lie on the same orbit have equal isotropy groups. On the other hand the isotropy groups of two points from different orbits are not equal and they are not conjugate because the group is commutative.

In the case when all orbits are of the same type, the cross-section $\mathcal{K}$ with the same isotropy group at each point, satisfies a more general definition of *slice* (see Bredon [4] definition 4.1 and theorem 4.4). For this reason we will call $\mathcal{K}$ an *isotypic slice*. When G is a compact Lie groups theorem 5.4 in [4] implies the existence of a slice. It can be generalized to the case of a proper action of a non-compact group [13].

Definition 3.2. The action of G is called *proper* if the map $\theta : G \times M \to M \times M$ defined by $\theta(g, z) = (g \cdot z, z)$ is proper. In other words if $K \subset M \times M$ is compact then so is $\theta^{-1}(K) \subset G \times M$.

It is not difficult to prove that any continuous action of a compact group is proper. Since $\theta^{-1}(z,z) = (H,z)$ where H is an isotropy group of z, if the action is proper, then the isotropy group of each point is compact. The following proposition can be deduced from theorems 4.4 and 0.9 in [4].

Proposition 3.3. *Let the action of G on M be proper with all orbits of the same type. Then a slice exists and is isotypic. In othere words a slice provides a local cross-section $\mathcal{K}$ with the same isotropy group at each point.*

We assume further that an isotypic slice exists at each order of prolongation. Our algorithm is based on the following observation.

Proposition 3.4. *Let G act regularly on a manifold M and let $\mathcal{K}$ be a slice. Then the condition $\rho(z)\cdot z \in \mathcal{K}$ for $z \in M$ defines a smooth G-equivariant map $[\rho] : M \to H\backslash G$.*

Proof. Let group elements g_1 and g_2 be such that $g_1 \cdot z \in \mathcal{K}$ and $g_2 \cdot z \in \mathcal{K}$. Since each orbit intersects the slice $\mathcal{K}$ at one point, then $g_1 \cdot z = g_2 \cdot z$ and so $g_2^{-1} g_1$ belongs to the isotropy group G_z of z. On the other hand, since H is the isotropy group of $g_2 \cdot z \in \mathcal{K}$, then $G_z = g_2^{-1} H g_2$. Thus $g_2^{-1} g_1 \in g_2^{-1} H g_2$. Hence $g_1 \in H g_2$, or equivalently $[g_1] = [g_2]$. We have proved that the map $[\rho]$ is well defined. The smoothness of $[\rho]$ follows from the smoothness of the group action and the smoothness of the standard projection $G \longrightarrow H\backslash G$.

To show the equivariance of $[\rho]$ we need to prove that $[\rho](g\cdot z) = [\rho](z) g^{-1}$. Let us choose $q_1 \in [\rho](z)$ and $q_2 \in [\rho](g \cdot z)$. By construction of $[\rho]$ one has

$$q_2 \cdot (g \cdot z) = q_1 \cdot z \in \mathcal{K}.$$

It follows that $q_1^{-1}\, q_2\, g \in G_z = q_1^{-1} H q_1$, or equivalently

$$q_2 \in H q_1 g^{-1}.$$

Since $[\rho](z) = Hq_1$ and $[\rho](g\cdot z) = Hq_2$ we have proved that $[\rho](g\cdot z) = [\rho](z) g^{-1}$. □

Note that we can extend $[\rho]$ to a G-equivariant map on J^k for any k by $[\rho](z^{(k)}) = [\rho]\left(\pi_0^k(z^{(k)})\right)$.

Algorithm 3.5. On *the zeroth step* we consider the action of the group G on M, such that there is an isotypic slice $\mathcal{K}_0$ with a constant isotropy group H_0. According to Proposition 3.4, the condition $\rho_0(z)\cdot z \in \mathcal{K}_0$ defines a smooth G-equivariant map $[\rho_0]$ from M to the right cosets $H_0\backslash G$. The non-constant coordinates of $\rho_0(z)\cdot z$ provide a complete set of the zeroth order invariants. The first prolongation of the H_0-action on J^1 can be restricted to the set $\mathcal{K}_0^1 = \{z^{(1)} | \pi_0^1(z^{(1)}) \in \mathcal{K}_0\} \subset J^1$.

On *the first step* we look for a cross-section $\mathcal{K}_1 \subset \mathcal{K}_0^1$ for the H_0-action on $\mathcal{K}_0^1$ with a constant isotropy group $H_1 \subset H_0$. We use this cross-section to define an H_0-equivariant map $[\tau_1] : \mathcal{K}_0^1 \to H_1\backslash H_0$, by the condition $\tau_1(z^{(1)}) \cdot z^{(1)} \in \mathcal{K}^1$ for $z^{(1)} \in \mathcal{K}_0^1$. Then the map

$$[\rho_1](z^{(1)}) = \left[\tau_1\left(\rho_0(z^{(1)})\cdot z^{(1)}\right)\rho_0(z^{(1)})\right] \tag{10}$$

is a G-equivariant map from J^1 to $H_1\backslash G$. Here τ_1 and ρ_0 denote representatives of the cosets $[\tau_1] \in H_1\backslash H_0$ and $[\rho_0] \in H_0\backslash G$ respectively, multiplied as elements of the group G. Since $\mathcal{K}^1$ is a isotypic slice for the first prolongation of the G-action with the isotropy group H_1, then $\rho_1(z) = \tau_1\left(\rho_0(z^{(1)})\cdot z^{(1)}\right)\rho_0(z^{(1)})$ satisfies the conditions of the Proposition 3.4 and thus the map $[\rho_1]$ is well defined (does not depend on the choice of representatives τ and ρ) and G-equivariant. One can also verify the correctness of definition (10) directly. It is clear that the coset $[\rho_1] \in H_1\backslash G$ does not depend on the choice the representative $\tau_1 \in H_1\backslash H_0$. Let $h_0\rho_0(z^{(1)})$, where $h_0 \in H_0$, be another representative of the coset $[\rho_0] \in H_0\backslash G$. Due to the H_0-equivariance of τ_1 we have

$$\begin{aligned} \tau_1\left(h_0\rho_0(z^{(1)})\cdot z^{(1)}\right)h_0\rho_0(z^{(1)}) &= \tau_1\left(\rho_0(z^{(1)})\cdot z^{(1)}\right)h_0^{-1}h_0\rho_0(z^{(1)}) \\ &= \tau_1\left(\rho_0(z^{(1)})\cdot z^{(1)}\right)\rho_0(z^{(1)}). \end{aligned}$$

The non-constant coordinates of $\rho_1(z^{(1)})\cdot z^{(1)}$ provide a complete set of the first order invariants.

We continue by prolonging the action of H_1 to the next order, or we may prolong by several orders at once if we wish. Thus we start *the k-th step* with a G-equivariant smooth map $[\rho_{k-1}] : J^{k-1} \longrightarrow H_{k-1}\backslash G$ which projects J^{k-1} to the cross-section $\mathcal{K}_{k-1}$ with a constant isotropy group H_{k-1}. We prolong the action of H_{k-1} to J^k and then restrict it to the H_{k-1}-invariant set $\mathcal{K}^k_{k-1} = \{z^{(k)}|\pi^k_{k-1}(z^{(k)}) \in \mathcal{K}_{k-1}\} \subset J^k$. For this restricted action of H_{k-1} we find an isotypic slice $\mathcal{K}_k \subset \mathcal{K}^k_{k-1}$ with a constant isotropy group H_k. This isotypic slice furnish a smooth H_{k-1} equivariant map $[\tau_k] : \mathcal{K}^k_{k-1} \longrightarrow H_k\backslash H_{k-1}$. The map

$$[\rho_k](z^{(k)}) = \left[\tau_k\left(\rho_{k-1}(z^{(k)})\cdot z^{(k)}\right)\rho_{k-1}(z^{(k)})\right] \tag{11}$$

is a G-equivariant map from J^k to $H_k\backslash G$. The non-constant coordinates of

$$\rho_k(z^{(k)})\cdot z^{(k)} \tag{12}$$

provide a complete set of k-th order differential invariants.

The algorithm terminates at the order where the isotropy group becomes trivial (or at least discrete). The next propostion shows that for regular jets this happens at the order n of stabilization.

Proposition 3.6. *Let n be the order of stabilization. Then on the n-the step of the above algorithm we will be able to find a local cross-section $\mathcal{K}_n$ with a discrete isotropy group.*

As before let $\mathcal{V}^n \subset J^n$ denote an open dense subset of regular jets. This means that the prolonged action is locally free on $\mathcal{V}^n$. In order to find a desired cross-section $\mathcal{K}_n$ it sufficient to show that the set $\mathcal{K}^n_{n-1} = \{z^{(n)}|\pi^n_{n-1}(z^{(n)}) \in \mathcal{K}_{n-1}\} \cap \mathcal{V}^n \neq \emptyset$. We first note that since all points on the same orbit have isomorphic isotropy groups, the sets $\mathcal{V}^n$ and $J^n\backslash\mathcal{V}^n$ are G-invariant. Thus, if the intersection above is empty, then $\mathcal{K}^n_{n-1} \subset J^n\backslash\mathcal{V}^n$ and hence $G\cdot\mathcal{K}^n_{n-1} = \{g\cdot z^{(n)}|z^{(n)} \in \mathcal{K}^n_{n-1}, g \in G\} \subset J^n\backslash\mathcal{V}^n$. On the other hand, $\pi^n_{n-1}\left(G\cdot\mathcal{K}^n_{n-1}\right) = G\cdot\left(\pi^n_{n-1}\mathcal{K}^n_{n-1}\right) = G\cdot\mathcal{K}_{n-1}$ is an open subset of J^{n-1}. Since π^n_{n-1} is continuous it follows that $G\cdot\mathcal{K}^n_{n-1}$ is an open subset of J^n that belongs to $J^n\backslash\mathcal{V}^n$. We arrive to a contradiction because $\mathcal{V}^n$ is dense. □

The algorithms resembles in many ways the one presented by M. Green [14] for constructing moving frames for curves in homogeneous spaces. However, taking advantage of the generalized approach by Fels and Olver [11], we can apply our algorithm to construct a moving frame for submanifolds of any dimension under more general (not necessarily transitive) group actions.

The algorithm produces the tower of the isotropy groups: $H_0 \supset H_1 \supset \cdots \supset H_n$, where H_n is at most discrete and the sequence of cross-section $\mathcal{K}_i$, $i = 1..n$, such that $\pi^k_{k-1}\mathcal{K}_k$ is an open dense subset of $\mathcal{K}_{k-1}$. Moreover, J^k splits into the product of the cross-section $\mathcal{K}_k$ and the homogeneous space $H_k\backslash G$. This splitting commutes with projections to the lower jets:

$$\begin{tikzcd}
J^n \arrow[r, "{([\rho_n], \iota_n)}"] \arrow[d, "\pi^n_{n-1}"'] & (H_n\backslash G) \times \mathcal{K}_n \arrow[d, "\delta^n_{n-1}"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
J^2 \arrow[r, "{([\rho_2], \iota_2)}"] \arrow[d, "\pi^2_1"'] & (H_2\backslash G) \times \mathcal{K}_2 \arrow[d, "\delta^2_1"] \\
J^1 \arrow[r, "{([\rho_1], \iota_1)}"] \arrow[d, "\pi^1_0"'] & (H_1\backslash G) \times \mathcal{K}_1 \arrow[d, "\delta^1_0"] \\
M \arrow[r, "{([\rho_0], \iota_0)}"'] & (H_0\backslash G) \times \mathcal{K}_0,
\end{tikzcd}$$

where π^k_{k-1} are the usual jet projections, the maps $\iota_k(z^{(k)}) = \rho_k(z^{(k)}) \cdot z^{(k)}$ and $\delta^k_{k-1}([g]_k, z^{(k)}) = \big([g]_{k-1}, \pi^k_{k-1}(z^{(k)})\big)$. Here by $[g]_k$ we mean the image of g under the standard projection $G \longrightarrow H_k\backslash G$. Since $H_{k-1} \supset H_k$ the maps δ are well defined.

It turns out that the above splitting induces a splitting of the space of invariant differential forms on J^k. Note that invariant differential forms on J^k form a module over the ring of k-th order differential invariants.

Proposition 3.7. *Let G be a group acting regularly on a manifold M, let $\mathcal{K}$ be an isotypic slice, that is a cross-section with the same isotropy group H at each point, and let $[\rho]$ be a G-equivariant map $M \longrightarrow H\backslash G$ defined by the condition $\iota(z) = \rho(z) \cdot z \in \mathcal{K}$, where $z \in M$ and $\rho(z)$ belongs to the coset $[\rho(z)]$. Then the module of invariant differential one-forms on M splits into a direct sum of the submodule $\Omega^G_{1,0} = \iota^*\left(T^*\mathcal{K}\right)$ of the pull-backs of differential one-forms on $\mathcal{K}$ under ι and submodule $\Omega^G_{0,1} = [\rho]^*\left(T^*H\backslash G\right)^G$ of the pull-backs of* invariant *differential one-forms on the homogeneous space $H\backslash G$ under $[\rho]$. This induces the following splitting in the space of invariant k forms: $\left(\bigwedge^k T^*M\right)^G = \bigoplus \Omega^G_{s,t}$, where $s+t = k, \quad s,t \geq 0$ and $\Omega^G_{s,t} = \bigwedge^s \Omega^G_{1,0} \bigotimes [\rho]^*\left(\bigwedge^t T^*H\backslash G\right)^G$.*

Proof. We first note that the map $\iota : M \longrightarrow \mathcal{K}$ is G-invariant and the map $[\rho] : M \longrightarrow H\backslash G$ is G-equivariant. Thus $\iota^*(\omega)$, where ω is *any* form on $\mathcal{K}$, and $[\rho]^*(\mu)$, where μ is an *invariant* form on $H\backslash G$, produce invariant forms on M. We conclude that forms in $\Omega^G_{1,0}$ and $\Omega^G_{0,1}$ are indeed invariant.

Let $\Omega_{1,0} = \iota^*\left(T^*\mathcal{K}\right)$ be the subspace of the cotangent bundle T^*M spanned over the ring of *all* differentiable functions on M by the forms $\iota^*(\omega)$, where $\omega \in T^*\mathcal{K}$. Similarly, let $\Omega_{0,1} = [\rho]^*\left(T^*H\backslash G\right)$ be the subspace of T^*M spanned $[\rho]^*(\mu)$, where μ is *any* one-form on $H\backslash G$. Note that ι is a projection from M onto $\mathcal{K}$, whose fibers are the orbits, so the restriction of any form from $\Omega_{1,0}$ to an orbit is zero. On the other hand, the map $[\rho]$ restricts to a local diffeomorphism between an orbit $\mathcal{O}$ and $H\backslash G$ and hence it induces ismorphism between the space of forms on $\mathcal{O}$ and on $H\backslash G$. Thus the restriction of the $[\rho]^*(\mu)$ to $\mathcal{O}$ is non-zero, unless $\mu \in T^*(H\backslash G)$ is a zero form. It follows that $\Omega_{1,0}$ and $\Omega_{0,1}$ have trivial intersection. By dimension count we conclude that $T^*M = \Omega_{1,0} \oplus \Omega_{0,1}$. Since $\Omega^G_{1,0}$ is a subset of $\Omega_{1,0}$ and $\Omega^G_{0,1}$ is a subset of $\Omega_{0,1}$ their intersection is also trivial. As a side remark, we notice that, since the map $[\rho]$ is constant along the leaves of the foliation $\{g \cdot \mathcal{K} | g \in G\}$, then the forms from $\Omega_{0,1}$ annihilate the tangent spaces to this foliation.

We now prove that any invariant form λ on M is in $\Omega^G_{1,0} \oplus \Omega^G_{0,1}$. Any one-form λ can be uniquely decomposed into two components $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 \in \Omega_{1,0}$ and $\lambda_2 \in \Omega_{0,1}$. Moreover, the subspaces $\Omega_{1,0}$ and $\Omega_{0,1}$ are preserved under the action of G because ι is G-invariant and $[\rho]$ is G-equivariant. Thus if λ is invariant, then $\lambda = g^*\lambda = g^*\lambda_1 + g^*\lambda_2$, where $g^*\lambda_1 \in \Omega_{1,0}$ and $g^*\lambda_2 \in \Omega_{0,1}$. From the uniqueness of the decomposition it follows that $g^*\lambda_1 = \lambda_1 \in \Omega^G_{1,0}$ and $g^*\lambda_2 = \lambda_2 \in \Omega^G_{0,1}$.

We are now turning to the description of the invariant k-forms on M. We first note that the G-invariant splitting of $T^*M = \Omega_{1,0} \oplus \Omega_{0,1}$ induces a G-invariant bigrading $\bigwedge T^*M = \bigoplus_{s,t\geq 0} \Omega_{s,t}$ of the algebra of differential forms on M, where $\Omega_{s,t} = (\bigwedge^s \Omega_{1,0}) \bigotimes \left(\bigwedge^t \Omega_{0,1}\right)$. Since the above bigrading is G-invariant, it induces a bigrading of the ring of invariant k-forms: $(\bigwedge T^*M)^G = \bigoplus_{s,t\geq 0} \Omega^G_{s,t}$, where $\Omega^G_{s,t} = (\bigwedge^s \Omega_{1,0})^G \bigotimes \left(\bigwedge^t \Omega_{0,1}\right)^G \subset \Omega_{s,t}$. Since $\Omega^G_{1,0}$ is spanned by the differentials of the complete set of independent invariant functions on M and has the same dimension over the ring of invariant functions as $\Omega_{1,0}$ over the ring of all functions on M, then $(\bigwedge^s \Omega_{1,0})^G = \bigwedge^s \Omega^G_{1,0}$. The space $\left(\bigwedge^t \Omega_{0,1}\right)^G$ is obtained by pulling back the space of G-invariant t forms on $H\backslash G$ under a G-invariant map $[\rho]$. □

Invariant forms on $H\backslash G$ are described by the following proposition in [23], ch. 13.

Proposition 3.8. *If $\pi\colon G \longrightarrow H\backslash G$ is the natural projection, then the map $\omega \longrightarrow \pi^*\omega$ is a one-to-one correspondence between the invariant k-forms on $H\backslash G$ and the right-invariant, $Ad(H)$ invariant k-forms on G, which annihilate the Lie algebra $\mathfrak{h}$ of H.*

Returning to the jet bundles J^k we can describe, in the case when the group action admits isotypic slices $\mathcal{K}_k$ with the isotropy group H_k, the structure of invariant differential forms, answering the question raised in [9]. Indeed, the maps ι_k and $[\rho_k]$ defined by (12) and (11) satisfy Proposition 3.7. Note also that once the action becomes locally free on J^n, then $H_n\backslash G$ islocally diffeomorphic to G and so admits an invariant coframe. The pull-back of this coframe under $[\rho_n]^*$ along with $\iota^*_n(T^*\mathcal{K}_n)$ form a local coframe on J^n in agreement with more general result of the existence of maximal number of relative invariants for locally free actions ([19], Theorem 3.36 and [9]).

Example 3.9. Let the special rotation group $G = SO(3,\mathbb{R})$ act on $M = \mathbb{R}^3\backslash\{0\} \times \mathbb{R}$ by rotations on the independent variables x, y, z and the trivial action on the dependent variable u.

The action of $SO(3,\mathbb{R})$ on $\mathbb{R}^3$ is not free. The isotropy group of every point on the positive half of the z-axis consists of the rotations around the z-axis. On the other hand, each orbit of $SO(3,\mathbb{R})$ intersects the positive half of the z-axis at a unique point, and hence it can serve as an isotypic slice $\mathcal{K}_0$ with the isotropy group $H_0 \backsimeq SO(2,\mathbb{R})$. We note that the quotient $H_0\backslash G = SO(3,\mathbb{R})\backslash SO(2,\mathbb{R})$ is diffeomorphic to a two dimensional sphere.

We start by constructing a map $[\rho_0] : M \to H_0\backslash G$ such that $\rho_0(\mathbf{z}) \cdot \mathbf{z} \in \mathcal{K}_0$ for any $\mathbf{z} \in M$. Each coset $G\backslash H_0$ can be represented as a product of two rotations: $R_x(\alpha)$ with respect to the x-axis and $R_y(\beta)$ with respect to the y-axis. In matrix form this can be written as

$$\begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta\sin\alpha & -\sin\beta\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{pmatrix}. \tag{13}$$

We choose the first rotation $R_x(\alpha)$ so that it brings an arbitrary point $\mathbf{z} = (x, y, z)$ to the upper xz-plane. It can be achieved by choosing the angle α, such that $\cos(\alpha) = \frac{z}{\sqrt{y^2+z^2}}$ and $\sin(\alpha) = \frac{y}{\sqrt{y^2+z^2}}$, with $0 \le \alpha \le \pi$, when $y \ge 0$ and $\pi < \alpha \le 2\pi$, when $y < 0$. Then $R_x(\alpha) \cdot \mathbf{z} = (x, 0, \sqrt{z^2+y^2})$. The latter point can moved to the positive z-axis by the rotation $R_y(\beta)$, where $\beta = \arctan(\frac{x}{\sqrt{z^2+y^2}})$. Then the only non-constant coordinate of $R_y(\beta)R_x(\alpha)\cdot\mathbf{z} = (0, 0, \sqrt{z^2+y^2+x^2})$ and the coordinate function u provide a fundamental set of zero order invariants. In the matrix form

$$\rho_0(\mathbf{z}) = R_y\, R_x = \begin{pmatrix} \frac{\sqrt{z^2+y^2}}{r} & -\frac{yx}{r\sqrt{z^2+y^2}} & -\frac{zx}{r\sqrt{z^2+y^2}} \\ 0 & \frac{z}{\sqrt{z^2+y^2}} & -\frac{y}{\sqrt{z^2+y^2}} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{pmatrix}, \tag{14}$$

where $r = \sqrt{z^2+y^2+x^2}$. We note that the above matrix provide a local description of the map $[\rho_0] : M \to H_0\backslash G$ on the coordinate chart which does not include the x-axis. On the complementary coordinate chart, that excludes the y-axis the same map is described by the matrix:

$$\begin{pmatrix} \frac{z}{\sqrt{x^2+z^2}} & 0 & -\frac{x}{\sqrt{x^2+z^2}} \\ -\frac{yx}{r\sqrt{x^2+z^2}} & \frac{\sqrt{x^2+z^2}}{r} & -\frac{yz}{r\sqrt{x^2+z^2}} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{pmatrix},$$

that belongs to the same coset of H_0 as (14).

Since it will be useful for next step, we compute the first jet coordinates of $\rho_0(\mathbf{z}^{(1)}) \cdot \mathbf{z}^{(1)}$ by prolonging the action (13) to the first order and then substituting elements of (14):

$$\begin{aligned} U_x &= \frac{\sqrt{z^2+y^2}}{r}u_x - \frac{yx}{r\sqrt{z^2+y^2}}u_y - \frac{zx}{r\sqrt{z^2+y^2}}u_z; \\ U_y &= \frac{z\,u_y - y\,u_z}{\sqrt{z^2+y^2}}; \\ U_z &= \frac{x}{r}u_x + \frac{y}{r}u_y + \frac{z}{r}u_z. \end{aligned} \tag{15}$$

On the next step we consider the action of the isotropy group H_0 on the set $\mathcal{K}_0^1 = \{\mathbf{z}^{(1)} | x = 0, y = 0, z > 0\} \subset J^1$. We note that on $\mathcal{K}_0^1$ the expressions r, u, U_x, U_y, U_z are equal to z, u, u_x, u_y, u_z repectively. Thus the former set can be considered as a coordinate set on $\mathcal{K}_0^1$ and will be used to describe the action of H_0

on $\mathcal{K}_0^1$:

$$
\begin{array}{lll} r \mapsto r; & u \mapsto u; & U_z \mapsto U_z; \\ U_x \mapsto \cos\gamma\, U_x - \sin\gamma\, U_y; & & U_y \mapsto \sin\gamma\, U_x + \cos\gamma\, U_y, \end{array} \tag{16}
$$

obtained by prolonging the action of

$$
\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

to the first order and then restricting it to $\mathcal{K}_0^1$. We observe that H_0 acts freely on $\mathcal{K}_0^1$ and choose the cross-section $\mathcal{K}^1 = \{\mathbf{z}^{(1)} \in \mathcal{K}_0^1 | U_x = 0\}$. This yields a map $\tau_1 \colon \mathcal{K}_0^1 \to H_0$ defined by the equation $\tan\gamma = \frac{U_x}{U_y}$ and the first order invariants:

$$
u, \quad r, \quad I_y = \sqrt{U_x^2 + U_y^2}, \quad I_z = U_z; \tag{17}
$$

Substitution of (15) into (17) is equivalent to computing the coordinates of

$$
\rho_1(\mathbf{z}^{(1)}) = \tau_1(\rho_0(\mathbf{z}^{(1)}) \cdot \mathbf{z}^{(1)})\rho_0(\mathbf{z}^{(1)}) \cdot \mathbf{z}^{(1)}
$$

and thus yields four fundamental first order invariants of $SO(3,\mathbb{R})$:

$$
\begin{array}{rl} \sqrt{z^2+y^2+x^2} = r, & u \\ \sqrt{(yu_z - zu_y)^2 + (zu_x - xu_z)^2 + (xu_y - yu_x)^2}, & \dfrac{x}{r}u_x + \dfrac{y}{r}u_y + \dfrac{z}{r}u_z. \end{array}
$$

The corresponding local moving frame $\rho_1 \colon J^1 \to SO(3,\mathbb{R})$, defined where I_y is non-zero, is given by the product

$$
\tau_1\,\rho_0 = \begin{pmatrix} \frac{U_y}{\sqrt{U_x^2+U_y^2}} & -\frac{U_x}{\sqrt{U_x^2+U_y^2}} & 0 \\ \frac{U_x}{\sqrt{U_x^2+U_y^2}} & \frac{U_y}{\sqrt{U_x^2+U_y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{z^2+y^2}}{r} & -\frac{yx}{r\sqrt{z^2+y^2}} & -\frac{zx}{r\sqrt{z^2+y^2}} \\ 0 & \frac{z}{\sqrt{z^2+y^2}} & -\frac{y}{\sqrt{z^2+y^2}} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{pmatrix},
$$

where U_x, U_y and r should be replaced by their expressions in terms of the initial coordinates.

Let us now look at the invariant differential forms, first on $M = \mathbb{R}^4$, and then on the jet spaces. From Proposition 3.7 we know that the module of invariant one-forms on M is a direct sum of the submodules Ω_1^G of the pull-backs of one-forms on $\mathcal{K}_0$ under the map $\iota_0(\mathbf{z}) = \rho_0(\mathbf{z}) \cdot \mathbf{z}$, and Ω_2^G of the pull-backs of *invariant* one-forms on $H_0 \backslash G$ under the map $[\rho_0]$. Thus Ω_1^G is spanned by dr and du which are the differentials of the fundamental zero order invariants. These forms actually span the whole module of invariant one-forms on M, because there is no invariant one forms on $H_0 \backslash G$, which is isomorphic to the two-dimensional sphere S^2, and so Ω_2^G is empty. There is however a non-trivial invariant two-form $\cos\beta\, d\beta \wedge d\alpha$ on S^2. Its pull-back under $[\rho_0]$ produces an invariant two-form $\omega = \frac{z\,dx\wedge dy + y\,dz\wedge dx + x\,dy\wedge dz}{r^3}$ on $M = \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}$, which along with the form $dr \wedge du$ spans the submodule of invariant two-forms on M. The basis for invariant three-forms is given by $\omega \wedge dr$, which is invariantly proportional to $dx \wedge dy \wedge dz$, and $\omega \wedge du$. Invariant four-forms are spanned by $\omega \wedge dr \wedge du$, which is invariantly proportional to $dx \wedge dy \wedge dz \wedge du$. The action is free on J^k, for all $k > 0$ and hence J^k admits an invariant coframe spanned by the differentials of fundamental invariants and the pull-back of invariant coframe on G.

4. A Moving Frame Construction for a Group that Factors as a Product.

We say that a group G factors as a product of its subgroups A and B if $G = A \cdot B$, that is, for any $g \in G$ there are $a \in A$ and $b \in B$ such that $g = ab$. We reproduce two useful statements from [13].

Theorem 4.1. *Let G be a group, and let A and B be two subgroups of G. Then the following conditions are equivalent:*
a) the reduction of the natural action of G on G/B to A is transitive,
b) $G = A \cdot B$,
c) $G = B \cdot A$,
d) the reduction of the natural action of G on G/A to B is transitive.

Corollary 4.2. *The reduction of the natural action of G on G/B to A is free and transitive if and only if $G = A \cdot B$ (or $G = B \cdot A$) and $A \cap B = e$.*

Remark 4.3. If $G = A \cdot B$ and $A \cap B = e$, then for each $g \in G$ there are *unique* elements $a \in A$ and $b \in B$ such that $g = ab$. In this case the manifolds $A \times B$ and G are diffeomorphic (although they are *not* in general isomorphic as groups). In the case when $A \cap B$ is discrete then $A \times B$ is locally diffeomorphic to G.

The following theorem plays a central role in our construction.

Theorem 4.4. *Let A and B act regularly on a manifold M, and assume that in a neighborhood U of a point $z_0 \in M$ the infinitesimal generators of the A-action are linearly independent from the generators of the B-action. Then locally there exists a submanifold $\mathcal{K}_A$ through the point z_0, which is transverse to the orbits of the subgroup A and is invariant under the action of the subgroup B.*

Proof. Let a be the dimension of the A-orbits, b be the dimension of the B-orbits on U and $m = \dim M$. By Frobenius' theorem we can locally rectify the orbits of B, that is, we can introduce coordinates $\{y_1, \dots, y_b, x_1, \dots, x_{m-b}\}$ on an open chart $U \subset M$ such that the intersection of the B-orbits with U are defined by the equations $x_i = k_i$, $i = 1, \dots, m-b$, where k_i are some constants. The orbits of B are integral manifolds for the distribution $\{\frac{\partial}{\partial y_1} \dots \frac{\partial}{\partial y_b}\}$. The functions x_i are invariant under the B-action. Let vector fields $X_1, \dots, X_a$ and $Y_1, \dots, Y_b$ be a basis for infinitesimal generators of the action of A and B respectively in a neighborhood U containing z_0 . The vector fields $Y_i, i = 1, \dots, b$ and $\frac{\partial}{\partial x_j}$, $j = 1 \dots m-b$ are linearly independent by the choice of coordinates, and their union forms a basis in TU. We can choose $c = m - b - a$ vector fields $\frac{\partial}{\partial x_{j_1}} \dots \frac{\partial}{\partial x_{j_c}}$ which are linearly independent from $X_1, \dots, X_a$ in TU. Let $\mathcal{K}$ be an integral manifold through the point z_0 for the involutive distribution $\Delta = \{\frac{\partial}{\partial x_{j_1}} \dots \frac{\partial}{\partial x_{j_c}}, \frac{\partial}{\partial y_1} \dots \frac{\partial}{\partial y_b}\}$. By construction $\mathcal{K}_A$ is a union of orbits of B and thus is invariant under the action of B. On the other hand, the distribution Δ is transverse to the infinitesimal generators $X_1, \dots, X_a$ of the A-action, and so is transverse to the orbits of A. $\square$

With this result we construct a moving frame for a product of groups A and B as follows.

Algorithm 4.5. Let $G = A \cdot B$ and let $A \cap B$ be discrete.

Then, as a manifold, G is locally diffeomorphic to $A \times B$. Let n be the order of stabilization of the G-action. Since both A and B act locally freely on $\mathcal{V}^n \subset J^n$ and their intersection is discrete then the infinitesimal generators of the A-action and the B-action are linearly independent at each point of $\mathcal{V}^n$ and hence they satisfy the conditions of Theorem 4.4. Thus there exists a local cross-section $\mathcal{K}_A^n \subset \mathcal{V}^n$ for the action of A which is invariant under the action of B. We use this cross-section to construct a moving frame ρ_A for A. The map $\rho_A(z^{(n)}) \cdot z^{(n)}$ projects $\mathcal{V}^n$ on the cross-section $\mathcal{K}_A^n$, which is invariant under the action of B. Moreover the action of B on $\mathcal{K}_A^n$ is locally free and hence we can choose a cross-section $\mathcal{K}^n \subset \mathcal{K}_A^n$ that defines a moving frame $\rho_B : \mathcal{K}_A^n \to B$. We can extend ρ_B to a map $\tilde{\rho}_B : \mathcal{V}^n \to B$ by the formula

$$\tilde{\rho}_B(z^{(n)}) = \rho_B\left(\rho_A(z^{(n)}) \cdot z^{(n)}\right). \tag{18}$$

The map $\tilde{\rho}_B$ is A-invariant but, in contrast to ρ_B, it is not B-equivariant. The cross-section $\mathcal{K}^n$ is transversal to the orbits of G and the map ρ_G defined by

$$\rho_G(z^{(n)}) = \tilde{\rho}_B\left(z^{(n)}\right)\rho_A(z^{(n)}) = \rho_B\left(\rho_A(z^{(n)}) \cdot z^{(n)}\right)\rho_A(z^{(n)}) \tag{19}$$

satisfies the condition $\rho_G(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}^n$, and hence is a moving frame for the G-action.

Remark 4.6. We emphasize that G-equivariance of the map ρ_G, claimed above, follows from the correspondence between cross-sections to the orbits of G and G-equivariant maps from J^n to G, discussed on page 2. On the other hand, it can be established explicitly using B-equivariance of the map ρ_B and the following lemma.

Lemma 4.7. *Let $G = A \cdot B$ act freely on a manifold N and let $\mathcal{K}_A$ be a local cross-section for the action of A, invariant under the B-action. Then the map $\rho_A : N \to A \subset G$ defined by the condition $\rho_A(z) \cdot z \in \mathcal{K}_A$ is G-equivariant up to the action of B, that is, for any $g \in G$ there exists $b \in B$ such that*

$$\rho_A(g \cdot z) = b\rho_A(z)g^{-1}.$$

Proof. Let $z_1 = \rho_A(z) \cdot z$ and $z_2 = \rho_A(g \cdot z)g \cdot z$. By the definition of ρ_A, both z_1 and z_2 belong to $\mathcal{K}_A$ and hence from the freeness of the action it follows that $\rho_A(z_1) = \rho_A(z_2) = e \in G$. Let

$$h = \rho_A(g \cdot z)g\rho_A(z)^{-1} \in G, \tag{20}$$

then $z_2 = h \cdot z_1$. Since $G = A \cdot B$, there exist $a \in A$ and $b \in B$, such that $h = ab$. Then

$$e = \rho_A(z_2) = \rho_A(ab \cdot z_1) = \rho_A(b \cdot z_1)a^{-1}. \tag{21}$$

The last equality follows from A-equivariance of ρ_A. On the other hand, $b \cdot z_1 \in \mathcal{K}_A$, since $\mathcal{K}_A$ is invariant under the action of B, and thus $\rho_A(b \cdot z_1) = e$. We conclude from (21) that $a = e$ and hence $h = b$. The lemma now follows from (20). □

The cross-section $\mathcal{K}_A^n$ and $\mathcal{K}^n$ and the maps ρ_A and ρ_B can be extended to higher order jet bundles as it was done in Section 2. The non-constant coordinate functions of

$$\rho_G(z^{(k)}) \cdot z^{(k)} = \rho_B\left(\rho_A(z^{(k)}) \cdot z^{(k)}\right)\rho_A(z^{(k)}) \cdot z^{(k)}, \quad k \geq n \tag{22}$$

provide a complete set of k-th order differential invariants for G.

Remark 4.8. We notice that the coordinates of $\rho_A(z^{(k)}) \cdot z^{(k)}$ are invariant under the A-action and thus the formula above expresses the invariants of the G-action in terms of the invariants of its subgroup A.

We can summarize our construction in the following commutative diagram, reminding the reader again that although all maps are written as if they were global, they might be defined only on open subsets of the manifolds appearing in (23).

(23)
$$\begin{tikzcd}
& & G \times J^\infty \arrow[d, shift left] \arrow[rrd, "w_G"] & & \\
J^\infty \arrow[rru, "\sigma_G"] \arrow[r, "\sigma_B"] & B \times J^\infty \arrow[r, "\sigma_A"] & B \times A \times J^\infty \arrow[u, shift left] \arrow[r, "w_A"] & B \times J^\infty \arrow[r, "w_B"] & J^\infty
\end{tikzcd}$$

where the maps w are defined by the prolonged group action:

$$\begin{aligned}
w_A(b, a, z^{(\infty)}) &= (b, a \cdot z^{(\infty)}), \\
w_B(b, z^{(\infty)}) &= b \cdot z^{(\infty)}, \\
w_G(g, z^{(\infty)}) &= g \cdot z^{(\infty)} = w_B \circ w_A(b, a, z^{(\infty)}) \text{ when } g = ba.
\end{aligned}$$

The maps σ are defined using the moving frames for A, B and G:

$$\begin{aligned}
\sigma_A(b, z^{(\infty)}) &= (b, \rho_A(z^{(\infty)}), z^{(\infty)}), \\
\sigma_B(z^{(\infty)}) &= (\tilde{\rho}_B(z^{(\infty)}), z^{(\infty)}) = \left(\rho_B\left(\rho_A(z^{(\infty)}) \cdot z^{(\infty)}\right), z^{(\infty)}\right), \\
\sigma_G(z^{(\infty)}) &= (\rho_G(z^{(\infty)}), z^{(\infty)}) = \sigma_A \circ \sigma_B(z^{(\infty)}).
\end{aligned}$$

The maps w_A and w_G are A-invariant, whence the maps σ_A and σ_B are A-equivariant, with respect to the A-actions on $B \times A \times J^\infty$, $B \times J^\infty$ and $G \times J^\infty$ defined respectively by:

$$\begin{aligned}
\tilde{a} \cdot (b, a, z^{(\infty)}) &= (b, a\tilde{a}^{-1}, \tilde{a} \cdot z^{(\infty)}), \\
\tilde{a} \cdot (b, z^{(\infty)}) &= (b, \tilde{a} \cdot z^{(\infty)}), \\
\tilde{a} \cdot (g, z^{(\infty)}) &= (g\tilde{a}^{-1}, \tilde{a} \cdot z^{(\infty)}).
\end{aligned}$$

The manifolds $B \times A \times J^k$ and $G \times J^k$ are locally diffeomorphic, and this diffeomorphism is A-equivariant. We note that neither σ_A nor σ_B are B-equivariant, but their composition is. As it has been discussed in the previous section (see formula (5)) the forms

$$\omega_G^i = \sigma_G^* \, d_H \, w_G^*(x^i), \; i = 1, \ldots, p, \tag{24}$$

produce a contact G-invariant coframe on J^∞. Since A is a subgroup of G then the forms ω_G^i retain their invariant properties under the action of A. On the other hand, the moving frame ρ_A provide us with another horizontal coframe which is contact invariant under the action of A:

$$\omega_A^i = \sigma_A^* \, d_H \, w_A^*(x^i), \; i = 1, \ldots, p.$$

These two coframes are related by a linear transformation $w_G^i = \sum_{j=1}^p L_j^i w_A^j$, where L_j^i are functions on J^∞ invariant under the A-action. In fact, L_j^i can be explicitly expressed in terms of the fundamental invariants of A:

$$\omega_G^i = \sigma_B^* \, \sigma_A^* \, d_H \, w_A^* \, w_B^*(x^i) = \sigma_B^* \sigma_A^* \pi_H w_A^* d_H \chi^i(b_1, \ldots, b_l, x^1, \ldots, x^p, u_J^\alpha), \tag{25}$$

where $\chi^i = w_B^* x^i$ is a function on $B \times J^\infty$, written in local coordinates $b_1, \ldots, b_l$, $x^1, \ldots, x^p$, u_J^α. The forms $\sigma_A^* \, \pi_H \, w_A^* \, d_H \chi^i$ are obtained from $d_H \chi^i$ by replacing forms dx^j with ω_A^j and coordinate functions $x^1, , \ldots, x^p, u_J^\alpha$ with their invariantizations $H^{(A)1}, , \ldots, H^{(A)p}, I_J^{(A)\alpha}$. These forms provide a horizontal coframe on $B \times J^\infty$

which is contact invariant with respect to the action of A. The final pull-back σ_B^* is equivalent to the replacement of parameters $b_1, ..., b_l$ with the corresponding coordinates of $\rho_B(\rho_A(z^{(\infty)}) \cdot z^{(\infty)})$. The latter are expressed in terms of invariants of the A-action.

In many situations the following reformulation of Theorem 4.1 enables us to enlarge a moving frame for a transformation group A to a moving frame for a larger group containing A.

Theorem 4.9. *Let $\mathcal{O} \subset M$ be an orbit of G and let A be a subgroup which acts transitively on $\mathcal{O}$. Then $G = B \cdot A$, where B is the isotropy group of a point in $\mathcal{O}$. If in addition A acts locally freely on $\mathcal{O}$ then $A \cap B$ is discrete.*

Let n_A be the order of stabilization for A, then the action of A is (locally) free on a subset $\mathcal{V}_A \subset J^{n_A}(M, p)$. Assume that the action of A can be extended to the action of a group G containing A, so that there is a point $z_0 \in \mathcal{V}_A$ such that the orbits of A and G through z_0 coincide. If this is the case, then let B be the isotropy group of the point z_0. Due to the theorem above $G = B \cdot A$ and $A \cap B$ is discrete, and so Algorithm 4.5 can be applied. An especially favorable case is when the action of A on the regular set $\mathcal{V}_A \subset J^{n_A}(M, p)$ is transitive. Then a moving frame for A can be extended to a moving frame for any group G containing A.

5. Examples: Euclidean, Affine and Projective Actions on the Plane.

The group of Euclidean motions on the plane is a factor of the group of special affine motions. In turn, the group of special affine motions is a factor of the group of projective transformations on the plane. Applying the Inductive Algorithm 4.5 we express projective invariants in terms of affine, and affine invariants in terms of Euclidean. We also obtain the relations among the Euclidean, affine and projective arc-lengths, and the corresponding invariant differential operators.

Example 5.1. Let us use the moving frame for the special Euclidean group $SE(2, \mathbb{R})$ acting on curves in $\mathbb{R}^2$ obtained in Example 2.7 to build a moving frame for the special affine group. We recall that the moving frame for $SE(2, \mathbb{R})$ has been obtained on the first jet space by choosing the cross-section $\{x = 0, u = 0, u_x = 0\}$. The special Euclidean group acts transitively on $J^1(\mathbb{R}^2, 1)$ and the first invariant, the Euclidean curvature κ, appears on the second order of prolongation. The normalization of u_{xxx} and u_{xxxx} yields the third and the fourth order invariants $I_3^e = \kappa_s$ and $I_4^e = \kappa_{ss} + 3\kappa^3$.

The special affine transformation $SA(2, \mathbb{R})$ on the plane is the semi-direct product of the special linear group $SL(2, \mathbb{R})$ and translations in $\mathbb{R}^2$. We prolong it to the first jet bundle and notice that the isotropy group B of the point $z_0^{(1)} = \{x = 0, u = 0, u_x = 0\}$ is given by all linear transformations of the form

$$\begin{pmatrix} \tau & \lambda \\ 0 & \frac{1}{\tau} \end{pmatrix}.$$

Due to Theorem 4.9 we obtain a product decomposition: $SA(2, \mathbb{R}) = B \cdot SE(2, \mathbb{R})$ and $B \cap SE(2, \mathbb{R})$ is finite. In fact $B \cap SE(2, \mathbb{R}) = \{\mathrm{I}, -\mathrm{I}\}$. Now we prolong the action of B up to fourth order:

$$\begin{aligned}
x &\to \tau x + \lambda u, \\
u &\to \frac{1}{\tau} u, \\
u_x &\to \frac{u_x}{\tau(\tau + \lambda u_x)}, \\
u_{xx} &\to \frac{u_{xx}}{(\tau + \lambda u_x)^3}, \\
u_{xxx} &\to \frac{(\tau + \lambda u_x) u_{xxx} - 3\lambda u_{xx}^2}{(\tau + \lambda u_x)^5}, \\
u_{xxxx} &\to \frac{(\tau + \lambda u_x)^2 u_{xxxx} - 10(\tau + \lambda u_x)\lambda u_{xx} u_{xxx} + 15\lambda^2 u_{xx}^3}{(\tau + \lambda u_x)^7}.
\end{aligned}$$

and restrict these transformations to the Euclidean cross-section $\mathcal{K}_E^4 = \{z^{(4)} | \pi_1^4(z^{(4)}) = z_0^{(1)}\} = \{z^{(4)} | x = 0, u = 0, u_x = 0\}$, obtaining

$$\begin{aligned}
u_{xx} &\to \frac{u_{xx}}{\tau^3}, \\
u_{xxx} &\to \frac{\tau u_{xxx} - 3\lambda u_{xx}^2}{\tau^5}, \\
u_{xxxx} &\to \frac{\tau^2 u_{xxxx} - 10\tau\lambda u_{xx} u_{xxx} + 15\lambda^2 u_{xx}^3}{\tau^7}.
\end{aligned}$$

The above action is free on the open subset $\{z^{(4)} \in \mathcal{K}_E^4 | u_{xx} \neq 0\}$, where we choose a cross-section

$$\mathcal{K}^{(4)} = \{z^{(4)} \in \mathcal{K}_E^4 | u_{xx} = 1, u_{xxx} = 0\}$$

to the orbits of B on $\mathcal{K}_E^4$. This produces a moving frame $\rho_B \colon \mathcal{K}_E^4 \to B$:

$$\tau = (u_{xx})^{1/3} \text{ and } \lambda = \frac{u_{xxx}}{3(u_{xx})^{5/3}}.$$

The corresponding fourth order invariant for the action of B on $\mathcal{K}_E^4$ is

$$I_4^b = \frac{u_{xx} u_{xxxx} - \frac{5}{3}(u_{xxx})^2}{(u_{xx})^{8/3}}. \tag{26}$$

We note that $\mathcal{K}^4$ can be viewed as a cross-section to the orbits of the entire group $SA(2, \mathbb{R})$ on the open subset of J^4 where $u_{xx} \neq 0$, and that due to formula (22) the fourth order special affine invariant can be obtained by invariantization of I_4^b with respect to Euclidean action, that is, by substitution of the normalized Euclidean invariants (9) into (26). Thus we obtain the expression of the lowest order special affine invariant μ in terms of Euclidean invariants:

$$\mu = I_4^a = \frac{I_2^e I_4^e - \frac{5}{3}(I_3^e)^2}{(I_2^e)^{8/3}}. \tag{27}$$

One can rewrite the normalized Euclidean invariants in terms of the Euclidean curvature and its derivatives: $I_2^e = \kappa$, $I_3^e = \kappa_s$ and $I_4^e = \kappa_{ss} + 3\kappa^3$, which leads to the expression:

$$\mu = \frac{\kappa(\kappa_{ss} + 3\kappa^3) - \frac{5}{3}\kappa_s^2}{\kappa^{8/3}}.$$

Remark 5.2. The reader might notice that the affine invariant obtained above differs by a factor of 3 from the classical *affine curvature* (see, for instance, [3] p.14). The appearance of this factor can be predicted from the recurrence formulae (13.4) in [11].

In accordance with (19) the moving frame for the special affine group corresponding to the cross-section $\mathcal{K}^4$ is the product of two matrices:

$$\begin{pmatrix} \kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\ 0 & \frac{1}{\kappa^{1/3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Using formula (25), one can obtain an affine contact invariant horizontal form $d\alpha$ in terms of the Euclidean arc-length ds:

$$d\alpha = \sigma_B^*\,\sigma_E^*\,\pi_H\,w_E^*\,d_H\,w_B^*\,(x),$$

where the Euclidean invariantization of $d_H\,w_B^*\,(x) = (\tau + \lambda u_x)\,dx$ equals to $\tau\,ds$ and hence

$$d\alpha = \sigma_B^*(\tau\,ds) = (I_2^e)^{1/3}ds = \kappa^{1/3}ds. \tag{28}$$

The form $d\alpha$ is called the affine arc-length. Written in the standard coordinates $d\alpha = u_{xx}^{1/3}dx$. The relation (28) between the affine and the Euclidean arc-lengths provide a natural explanation for the affine curve evolution equation in [22]. The relation between invariant differential operators follows immediately:

$$\frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}}\frac{d}{ds},$$

which enables us to obtain all higher order affine invariants in terms of the Euclidean ones.

Example 5.3. Let us now use the moving frame for the special affine group to build a moving frame for the projective group $PSL(3,\mathbb{R})$, whose local action on the plane is given by the transformations:

$$\begin{array}{rcl} x & \mapsto & \dfrac{\alpha x + \beta u + \gamma}{\delta x + \epsilon u + \zeta}, \\ u & \mapsto & \dfrac{\lambda x + \nu u + \tau}{\delta x + \epsilon u + \zeta}, \end{array}$$

where the determinant of the corresponding 3×3 matrix equals to one. The affine moving frame above corresponds to the cross-section

$$z_0^{(3)} = \{x=0, u=0, u_1=0, u_2=1, u_3=0\} \in J^3.$$

The isotropy group B of $z_0^{(3)}$ for the prolonged action of $PSL(3,\mathbb{R})$ consists of the transformations:

$$\begin{pmatrix} 1 & ab & 0 \\ 0 & a & 0 \\ b & c & \frac{1}{a} \end{pmatrix}.$$

Due to Theorem 4.9 we obtain a product decomposition: $PSL(3,\mathbb{R}) = B\cdot SA(2,\mathbb{R})$ and $B\cap SA(2,\mathbb{R})$ is finite. The affine cross-section

$$\mathcal{K}_A^7 = \{z^{(7)} | \pi_3^7(z^{(7)}) = z_0^{(3)}\} = \{z^{(7)} |\, x=0, u=0, u_x=0, u_{xx}=1, u_{xxx}=0\}$$

is invariant under the action of B. The seventh order prolongation of the B-action on J^7 has been computed using the MAPLE package VESSIOT [1]. The restriction of this action to $\mathcal{K}^7_A$ is given by

$$\begin{aligned}
u_4 &\to \frac{u_4 - 3a^2b^2 + 6ac}{a^2},\\
u_5 &\to \frac{u_5}{a^3},\\
u_6 &\to \frac{u_6 + 3abu_5 + 30u_4(2ac - a^2b^2) + 180a^2c(c - ab^2) + 45a^2b^2}{a^4},\\
u_7 &\to \frac{u_7 + 7abu_6 + u_5(105ac - 42b^2a^2) - 35(u_4)^2ab}{a^5}.
\end{aligned}$$

The above action is free on the open subset $\{z^{(7)} \in \mathcal{K}^7_A | u_5 \neq 0\}$, where we choose a cross-section

$$\mathcal{K}^7 = \{z^{(7)} \in \mathcal{K}^7_A | u_4 = 0, u_5 = 1, u_6 = 0\}$$

to the orbits of B on $\mathcal{K}^7_A$. This produces a moving frame $\rho_B \colon \mathcal{K}^7_A \to B$ given by

$$\begin{aligned}
a &= (u_5)^{1/3},\\
b &= \frac{5(u_4)^2 - u_6}{3(u_5)^{4/3}},\\
c &= \frac{(u^6)^2 - 10u_6(u_4)^2 - 3u_4(u_5)^2 + 25(u_4)^4}{18(u_5)^{7/3}}.
\end{aligned}$$

The corresponding seventh order differential invariant (for the B-action on $\mathcal{K}^7_A$) is

$$I^b_7 = \frac{6u_7u_5 - 7(u_6)^2 + 70(u_4)^2u_6 - 105u_4(u_5)^2 - 175(u_4)^4}{6(u_5)^{8/3}}. \tag{29}$$

We note that $\mathcal{K}^7$ can be viewed as a cross-section to the orbits of the entire group $PSL(3,\mathbb{R})$ on the open subset J^7, where $u_5 \neq 0$. Due to formula (22) the lowest order projective invariant can be obtained by invariantization of I^b_7 with respect to the affine action, that is, by substitution of the normalized affine invariants $I^a_4 = \mu$, $I^a_5 I^a_6$ and I^a_7 in (29). Note that we do not need the explicit formulae for these invariants. Thus we obtain a seventh order projective invariant η in terms of the special affine invariants:

$$\eta = I^p_7 = \frac{6I^a_7 I^a_5 - 7(I^a_6)^2 + 70(I^a_4)^2 I^a_6 - 105 I^a_4 (I^a_5)^2 - 175(I^a_4)^4}{6(I^a_5)^{8/3}}.$$

Using the recursion algorithm from [11] we can express the higher order affine invariants in terms of μ and its derivatives with respect to *affine* arc-length $d\alpha = u_{xx}^{1/3}dx$.

Thus

$$\begin{aligned}
I^a_4 &= \mu, & I^a_5 &= \mu_\alpha,\\
I^a_6 &= \mu_{\alpha\alpha} + 5\mu^2, & I^a_7 &= \mu_{\alpha\alpha\alpha} + 17\mu\mu_\alpha.
\end{aligned}$$

This leads to the formula:

$$\eta = \frac{-7\mu^2_{\alpha\alpha} + 6\mu_\alpha\mu_{\alpha\alpha\alpha} - 3\mu\mu^2_\alpha}{6\mu^{8/3}_\alpha}.$$

Remark 5.4. The above expression can be compared with analogous formula (61) in [8]. Keeping in mind that the affine curvature used there differs from μ by a factor $\frac{1}{3}$, we notice that η differs from the classical projective curvature by a factor of $6^{-5/3}$.

The moving frame for the projective group is the product of the matrices:

$$\begin{pmatrix} 1 & -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_\alpha} & 0 \\ 0 & \mu_\alpha^{1/3} & 0 \\ -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_\alpha^{4/3}} & \frac{1}{18}\frac{\mu_{\alpha\alpha}^2-3\mu\mu_\alpha^2}{\mu_\alpha^{7/3}} & \frac{1}{\mu_\alpha^{1/3}} \end{pmatrix} \begin{pmatrix} \kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\ 0 & \frac{1}{\kappa^{1/3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{u u_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{x u_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

We can express the projective arc-length (that is, a horizontal form which is contact invariant with respect to the projective action) in terms of the affine arc-length $d\alpha$. We first lift the coordinate function x to $B\times J^\infty$ by $w_B^*(x) = \frac{x+abu}{bx+cu+\frac{1}{a}}$. The affine invariantization of $d_H\, w_B^*(x)$ produces a horizontal form $a\,d\alpha$ on $B\times J^\infty$, where $d\alpha$ is the affine arc-length (28). The projective arc-length equals to

$$d\varrho = \sigma_B^*\, a\, d\alpha = (I_5^a)^{1/3} d\alpha = \mu_\alpha^{1/3} d\alpha.$$

The relation between invariant derivatives, $\frac{d}{d\varrho} = \frac{1}{\mu_\alpha^{1/3}}\frac{d}{d\alpha}$, allows us to obtain all higher order projective invariants in terms of the affine ones.

6. Euclidean and Affine Actions on Curves in $\mathbb{R}^3$.

More generally, the Euclidean group $SE(n,\mathbb{R})$ of orientation preserving rigid motions in $\mathbb{R}^n$ is a factor of the n-dimensional special affine group $SA(n,\mathbb{R})$. In turn, the special affine group is a factor of the projective group $PSL(n+1,\mathbb{R})$. Thus, the above procedure can be carried out in higher dimensions (computations become quite difficult, however). In this section we use the moving frames and invariants for the Euclidean action on curves in $\mathbb{R}^3$ to build up the moving frame and differential invariants for the affine action.

Example 6.1. The classical (left) moving frame for a curve in $\mathbb{R}^3$ consists of a point on a curve with coordinates (x,u,v), the tangent vector at this point, the normal and the binormal vectors. It is not difficult to verify that the cross-section $\mathcal{K}_E^2 = \{z^{(2)} | x = 0, u = 0, v = 0, u_x = 0, v_x = 0, v_{xx} = 0\} \subset J^2$ gives rise to the corresponding right moving frame. The latter produces normalized differential invariants, which can be rewritten in terms of the classical Euclidean curvature and torsion, and their derivatives with respect to the arc-length ds, using recurrence formulae [11] :

$$\begin{aligned} I_2^{u,e} &= \kappa, \\ I_3^{u,e} &= \kappa_s, \\ I_3^{v,e} &= \kappa\,\tau, \\ I_4^{u,e} &= \kappa_{ss} + 3\,\kappa^3 - \kappa\,\tau^2, \\ I_4^{v,e} &= 2\,\kappa_s\,\tau + \kappa\,\tau_s, \\ I_5^{u,e} &= \kappa_{sss} + 19\,\kappa_s\,\kappa^2 - 3\,\tau^2\,\kappa_s - 3\,\kappa\,\tau\,\tau_s, \\ I_5^{v,e} &= \tau_{ss}\,\kappa + 3\,\tau\,\kappa_{ss} + 9\,\kappa^3\,\tau - \kappa\,\tau^3 + 3\,\kappa_s\,\tau_s. \end{aligned} \tag{30}$$

The isotropy group B of $\mathcal{K}^2_E$ consists of upper triangular special linear transformations:

$$\begin{pmatrix} a & b & c \\ 0 & f & g \\ 0 & 0 & \frac{1}{a\,f} \end{pmatrix}.$$

Due to Theorem 4.9 we obtain a product decomposition: $SA(3,\mathbb{R}) = B \cdot SE(3,\mathbb{R})$ and $B \cap SA(3,\mathbb{R}) = I$. We prolong the action of B to the fifth order and then restrict it to the Euclidean cross-section $\mathcal{K}^5_E = \{z^{(5)} | \, x = 0, u = 0, v = 0, u_x = 0, v_x = 0, v_{xx} = 0\} \subset J^5$, which is invariant under the action of B. The action of B on $\mathcal{K}^5_E$ is given by

$$\begin{aligned}
u_2 &\to \frac{f\,u_2}{a^2},\\
u_3 &\to \frac{f}{a^5}\left(a^2\,u_3 + g\,a^2\,v_3 - 3\,b\,f\,a\,u_2^2\right),\\
v_3 &\to \frac{v_3}{a^4\,f},\\
u_4 &\to \frac{1}{a^7}\left(-4\,u_2\,v_3\,c\,f\,a^2 + u_4\,f\,a^3 + v_4\,g\,a^3 + 15\,b^2\,u_2^3\,f\,a - 10\,u_2\,u_3\,f\,a^2\right.\\
&\qquad \left.-6\,u_2\,v_3\,g\,a^2\,b\right),\\
v_4 &\to \frac{1}{a^8\,f}\left(-6\,u_2\,v_3\,a^2\,b + v_4\,a^3\right),\\
u_5 &\to \frac{1}{a^8\,f}\left(45\,b^2\,u_2^2\,v_3 - 10\,b\,u_2\,v_4\,a - 10\,b\,v_3\,u_3\,a - 10\,v_3^2\,a\,c + v_5\,a^2\right),\\
v_5 &\to \frac{1}{a^8}\left(45\,u_2^2\,v_3\,g\,a\,b^2 + 105\,u_2^2\,u_3\,a\,f\,b^2 + 60\,u_2^2\,v_3\,c\,f\,a\,b - 15\,b\,u_2\,f\,u_4\,a^2\right.\\
&\qquad -10\,u_2\,v_4\,g\,a^2\,b - 5\,u_2\,v_4\,c\,f\,a^2 - 105\,b^3\,u_2^4\,f - 10\,u_3\,a^2\,v_3\,g\,b\\
&\qquad \left.-10\,u_3^2\,a^2\,f\,b - 10\,u_3\,a^2\,v_3\,c\,f - 10\,v_3^2\,a^2\,g\,c + f\,u_5\,a^3 + g\,v_5\,a^3\right).
\end{aligned}$$

The above action is free on an open subset $\{z^{(5)} \in \mathcal{K}^5_E | u_2\,v_3 \neq 0\}$. On the open subset, where $u_2\,v_3 > 0$ one can choose the cross-section

$$\mathcal{K}^5 = \{z^{(5)} \in \mathcal{K}^7_A | u_2 = 1, u_3 = 0, v_3 = 1, u_4 = 0, v_4 = 0\}$$

to the orbits of B on $\mathcal{K}^5_E$. Assuming further that $u_2\,v_3 > 0$ (equivalently $\tau > 0$), we note, however, that the cross-section $\{z^{(5)} \in \mathcal{K}^7_A | u_2 = 1, u_3 = 0, v_3 = -1, u_4 = 0, v_4 = 0\}$ can be chosen on the subset where $u_2\,v_3 < 0$ and similar computations can be conducted.

The cross-section $\mathcal{K}^5$ leads to the moving frame $\rho_B \colon \mathcal{K}^5_E \to B$, given by

$$\begin{aligned}
a &= (u_2\,v_3)^{1/6},\\
b &= \frac{1}{6}\frac{v_4}{(u_2\,v_3)^{5/6}},\\
c &= \frac{1}{48}\frac{(v_2\,u_3)^{1/6}(12\,u_4\,u_3^2 + 5\,v_4^2\,u_2 - 20\,u_3\,v_3\,v_4)}{u_2\,v_3^3},\\
f &= \frac{v_3}{(u_2\,v_3)^{2/3}},\\
g &= \frac{1}{2}\frac{-2\,u_3\,v_3 + v_4\,u_2}{v_3(u_2\,v_3)^{2/3}}.
\end{aligned}$$

The corresponding fifth order differential invariants (for the B-action on $\mathcal{K}^5_E$) are found by substitution of the group parameters in the transformation formulae for u_5 and v_5.

$$I_5^{u,b} = \frac{90\, v_4\, v_3 (u_3\, v_4 - v_3\, u_4) - 25\, v_4^3\, u_2 + 36\, v_3^2\, (v_3\, u_5 - u_3\, v_5) + 18\, v_5 v_3 u_2 v_4}{36\, u_2^{3/2}\, v_3^{7/2}},$$

$$I_5^{v,b} = \frac{-35\, v_4^2\, u_2 + 60\, u_3\, v_4\, v_3 - 60\, u_4\, v_3^2 + 24\, v_5\, v_3\, u_2}{24\, u_2^{4/3}\, v_3^{7/3}}. \tag{31}$$

We note that $\mathcal{K}^5$ can be viewed as a cross-section to the orbits of the entire group $SA(3,\mathbb{R})$ on an open subset J^5, where $u_2\, v_3 > 0$. Due to formula (22) the two lowest order affine invariants can be obtained by invariantization of $I_5^{u,b}$ and $I_5^{v,b}$ with respect to Euclidean action, that is, by substitution of the normalized Euclidean invariants $I_j^{u,e}$ for u_j and $I_j^{v,e}$ for v_j, $\quad j = 1, \ldots, 5$ in (31). Using formulae (30), we obtain two fifth order affine invariant $I_5^{u,a}$ and $I_5^{v,a}$ in terms of the Euclidean curvature, torsion and their derivatives with respect to the Euclidean arc-length ds:

$$\begin{aligned} I_5^{u,a} \;&=\; \frac{1}{36}\,\kappa^{-4}\tau^{-7/2}\,\big(36\kappa^2\tau^2(\tau\kappa_{sss} - \kappa_{ss}\tau_s + 4\kappa^2\tau\kappa_s - \kappa\tau^2\tau_s - 3\kappa^3\tau_s + 2\tau^3\kappa_s) \\ &+\; 60\tau^2\kappa(\kappa_s^2\tau_s - 3\tau\kappa_{ss}\kappa) - 6\kappa^2\tau(\tau_s^2\kappa_s - 3\kappa\tau_{ss}\tau_s) + 160\kappa_s^3\tau^3 - 25\kappa^3\tau_s^3\big)\,, \\ I_5^{v,a} \;&=\; \frac{36\kappa^2\tau^2(\kappa^2+\tau^2) - 20\kappa_s^2\tau^2 - 8\kappa\tau\kappa_s\tau_s - 35\kappa^2\tau_s^2 + 12\kappa\tau(\tau\kappa_{ss} + 2\kappa\tau_{ss})}{24\,\kappa^{8/3}\,\tau^{7/3}}. \end{aligned}$$

As before, one can express the affine arc-length in terms of the Euclidean arc-length:

$$d\alpha = \sigma_B^* a\, ds = (I_2^{u,e}\, I_3^{v,e})^{1/6}\, ds = (\kappa^2\, \tau)^{1/6}\, ds,$$

and thus obtain all higher order affine invariants in terms of the Euclidean ones.

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