Algorithms from the paper

Object-image correspondence for curves under central and parallel projections[*]

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1 Formulas for invariants

$$\Delta_1 = 3\,y^{(4)}\,y^{(2)} - 5\,[y^{(3)}]^2 \text{ and } \Delta_2 = 9\,y^{(5)}\,[y^{(2)}]^2 - 45\,y^{(4)}\,y^{(3)}\,y^{(2)} + 40\,[y^{(3)}]^3. \tag{1}$$

Differentially separating set of rational $\mathcal{A}(2)$-invariants:

$$\begin{aligned} K_{\mathcal{A}} &= \frac{(\Delta_2)^2}{(\Delta_1)^3}; \\ T_{\mathcal{A}} &= \frac{9\,y^{(6)}\,[y^{(2)}]^3 - 63\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2 - 45\,[y^{(4)}]^2\,[y^{(2)}]^2 + 255\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} - 160\,[y^{(3)}]^4}{(\Delta_1)^2}. \end{aligned} \tag{2}$$

Differentially separating set of rational $\mathcal{PGL}(3)$-invariants:

$$\begin{aligned} K_{\mathcal{P}} &= \frac{729}{8\,(\Delta_2)^8}\Big(18\,y^{(7)}\,[y^{(2)}]^4\;\Delta_2 - 189\,[y^{(6)}]^2\,[y^{(2)}]^6 \\ &+ 126\,y^{(6)}\,[y^{(2)}]^4\;(9\,y^{(5)}\,y^{(3)}\,y^{(2)} + 15\,[y^{(4)}]^2\,y^{(2)} - 25\,y^{(4)}\,[y^{(3)}]^2) \\ &- 189\,[y^{(5)}]^2\,[y^{(2)}]^4\;(4\,[y^{(3)}]^2 + 15\,y^{(2)}\,y^{(4)}) \\ &+ 210\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2\;(63\,[y^{(4)}]^2\,[y^{(2)}]^2 - 60\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} + 32\,[y^{(3)}]^4) \\ &- 525\,y^{(4)}y^{(2)}\;(9\,[y^{(4)}]^3\,[y^{(2)}]^3 + 15\,[y^{(4)}]^2\,[y^{(3)}]^2\,[y^{(2)}]^2 - 60\,y^{(4)}\,[y^{(3)}]^4\,y^{(2)} + 64\,[y^{(3)}]^6) \\ &+ 11200\,[y^{(3)}]^8\Big)^3; \\ T_{\mathcal{P}} &= \frac{243\,[y^{(2)}]^4}{2\,(\Delta_2)^4}\;\Big(2\,y^{(8)}\,y^{(2)}\,(\Delta_2)^2 \\ &- 8\,y^{(7)}\,\Delta_2\,(9\,y^{(6)}\,[y^{(2)}]^3 - 36\,y^{(5)}\,y^{(3)}\,[y^{(2)}]^2 - 45\,[y^{(4)}]^2\,[y^{(2)}]^2 + 120\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)} - 40\,[y^{(3)}]^4) \\ &+ 504\,[y^{(6)}]^3\,[y^{(2)}]^5 - 504\,[y^{(6)}]^2\,[y^{(2)}]^3\;(9\,y^{(5)}\,y^{(3)}\,y^{(2)} + 15\,[y^{(4)}]^2\,y^{(2)} - 25\,y^{(4)}\,[y^{(3)}]^2) \\ &+ 28\,y^{(6)}\;\big(432\,[y^{(5)}]^2\,[y^{(3)}]^2\,[y^{(2)}]^3 + 243\,[y^{(5)}]^2\,y^{(4)}\,[y^{(2)}]^4 - 1800\,y^{(5)}\,y^{(4)}\,[y^{(3)}]^3\,[y^{(2)}]^2 \\ &- 240y^{(5)}\,[y^{(3)}]^5\,y^{(2)} + 540y^{(5)}\,[y^{(4)}]^2\,[y^{(3)}]\,[y^{(2)}]^3 + 6600\,[y^{(4)}]^2\,[y^{(3)}]^4\,y^{(2)} - 2000y^{(4)}\,[y^{(3)}]^6 \\ &- 5175\,[y^{(4)}]^3\,[y^{(3)}]^2\,[y^{(2)}]^2 + 1350\,[y^{(4)}]^4\,[y^{(2)}]^3\big) - 2835\,[y^{(5)}]^4\,[y^{(2)}]^4 \\ &+ 252\,[y^{(5)}]^3y^{(3)}\,[y^{(2)}]^2\;(9y^{(4)}\,y^{(2)} - 136\,[y^{(3)}]^2) - 35840\,[y^{(5)}]^2\,[y^{(3)}]^6 \\ &- 630\,[y^{(5)}]^2\,[y^{(4)}]\,[y^{(2)}]\;(69\,[y^{(4)}]^2\,[y^{(2)}]^2 - 160\,[y^{(3)}]^4 - 153\,y^{(4)}\,[y^{(3)}]^2\,[y^{(2)}]) \\ &+ 2100\,y^{(5)}\,[y^{(4)}]^2\,y^{(3)}\;(72\,[y^{(3)}]^4 + 63\,[y^{(4)}]^2\,[y^{(2)}]^2 - 193\,y^{(4)}\,[y^{(3)}]^2\,y^{(2)}) \\ &- 7875\,[y^{(4)}]^4\;(8\,[y^{(4)}]^2\,[y^{(2)}]^2 - 22y^{(4)}\,[y^{(3)}]^2\,[y^{(2)}] + 9\,[y^{(3)}]^4)\Big). \end{aligned} \tag{3}$$

Remark 1. *If $\mathcal{X}$ is a parametric curve $(x(t), y(t))$, then the restriction of the above invariants to $\mathcal{X}$ is computed using*

$$y^{(1)} = \frac{\dot{y}}{\dot{x}} \text{ and } y^{(k)} = \frac{\dot{y}^{(k-1)}}{\dot{x}},$$

where $\dot{}$ here and below denotes the derivative with respect to the parameter. *If $\mathcal{X}$ is given by an* *implicit equation* $F(x,y)=0$ *then the restriction of the above invariants to* $\mathcal{X}$ *is computed using*

$$\begin{aligned} y^{(1)} &= -\frac{F_x}{F_y}, \\ y^{(2)} &= \frac{-F_{xx}\,F_y^2 + 2\,F_{xy}\,F_x\,F_y - F_{yy}\,F_x^2}{F_y^3}, \\ &\vdots \end{aligned}$$

Remark 2. *If* Δ_1 *is zero at more than a finite number of points of an algebraic curve* $\mathcal{X}$*, then* $\mathcal{X}$ *is either a line or a parabola, and then* $\Delta_1 \equiv 0$*. If* Δ_2 *is zero at more than a finite number of points of* $\mathcal{X}$*, then* $\mathcal{X}$ *is either a line or a conic, and then* $\Delta_2 \equiv 0$*.*

2 Algorithms

2.1 Central projections

Algorithm 3. (Central projections for parametric algebraic curves.)

INPUT: Rational maps $\Gamma(s) = \big(z_1(s), z_2(s), z_3(s)\big)$*,* $s \in \mathbb{R}$ *and* $\gamma(t) = (x(t),\, y(t))$*,* $t \in \mathbb{R}$ *parametrizing algebraic curves* $\mathcal{Z} \subset \mathbb{R}^3$ *and* $\mathcal{X} \subset \mathbb{R}^2$*.*

OUTPUT: The truth of the statement: $\exists [P] \in \mathcal{CP},$ *such that* $\mathcal{X} = P(\mathcal{Z})$*.*

STEPS:

1. *evaluate* $\Delta_2|_{\mathcal{X}}$ *using (1). If* $\Delta_2|_{\mathcal{X}}(t) \equiv 0$ *follow Algorithm 5 else*
2. *evaluate* $\mathcal{PGL}(3)$*-invariants (3) on* $\mathcal{X}$*. Obtain two rational functions* $K_{\mathcal{P}}|_{\mathcal{X}}$ *and* $T_{\mathcal{P}}|_{\mathcal{X}}$ *of* t*;*
3. *for arbitrary* $c_1, c_2, c_3 \in \mathbb{R}$ *define a curve* $\tilde{\mathcal{Z}}_c$ *parametrized by* $\epsilon_c(s) = \left(\frac{z_1(s)+c_1}{z_3(s)+c_3},\, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$*;*
4. *evaluate* $\mathcal{PGL}(3)$*-invariants (3) on* $\tilde{\mathcal{Z}}_c$*. Obtain two rational functions* $K_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}$ *and* $T_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}$ *of* c *and* s*;*
5. *determine the truth of the statement:*

$$\exists c \in \mathbb{R}^3 \quad \forall s \in \mathbb{R} \quad \exists t \in \mathbb{R} \quad K_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}(c,\, s) = K_{\mathcal{P}}|_{\mathcal{X}}(t) \text{ and } T_{\mathcal{P}}|_{\tilde{\mathcal{Z}}_c}(c,\, s) = T_{\mathcal{P}}|_{\mathcal{X}}(t).$$

Remark 4. *The last step of Algorithm 3 is a computationally challenging real quantifier eliminations problem. Algorithmic solution of such problems implemented, for instance, in* Mathematica *(see command "Reduce"). Our* Maple *implementation gives a solution to this problem over the complex numbers. The truth of the statement over the complex numbers provides a necessary but not a sufficient condition for the truth over the real numbers.*

Algorithm 5. (Central projections to lines and conics.)

INPUT: Rational maps $\Gamma(s) = \big(z_1(s), z_2(s), z_3(s)\big)$*,* $s \in \mathbb{R}$ *and* $\gamma(t) = (x(t),\, y(t))$*,* $t \in \mathbb{R}$ *parametrizing algebraic curves* $\mathcal{Z} \subset \mathbb{R}^3$ *and* $\mathcal{X} \subset \mathbb{R}^2$*, such that* $\Delta_2|_{\mathcal{X}}(t) \equiv 0$*.*

OUTPUT: The truth of the statement: $\exists [P] \in \mathcal{CP},$ *such that* $\mathcal{X} = P(\mathcal{Z})$*.*

STEPS:

1. *if* $\ddot{x}\,\dot{y} - \ddot{y}\,\dot{x} \equiv 0$ *and* $(\ddot{\Gamma} \times \dot{\Gamma}) \cdot \dddot{\Gamma} \equiv 0$ *return TRUE and exit;*
2. *if* $\ddot{x}\,\dot{y} - \ddot{y}\,\dot{x} \equiv 0$ *and* $(\ddot{\Gamma} \times \dot{\Gamma}) \cdot \dddot{\Gamma} \neq 0$ *return FALSE and exit;*
3. *for arbitrary* $c_1, c_2, c_3 \in \mathbb{R}$ *define a curve* $\tilde{\mathcal{Z}}_c$ *parametrized by* $\epsilon_c(s) = \left(\frac{z_1(s)+c_1}{z_3(s)+c_3},\, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$;
4. *evaluate* $\Delta_2|_{\tilde{\mathcal{Z}}_c}$ *using (1). Obtain a rational function* $\Delta_2|_{\tilde{\mathcal{Z}}_c}$ *of* c *and* s.
5. *determine the truth of the statement:* $\exists c \in \mathbb{R}^3 \quad \forall s \in \mathbb{R} \quad \Delta_2|_{\tilde{\mathcal{Z}}_c}(c, s) = 0.$

Remark 6. *The first two steps of the above algorithm rely on the fact that a space curve can be projected to a line if and only if the space curve is coplanar.*

2.2 Parallel projections

Algorithm 7. (Parallel projections for parametric algebraic curves.)
INPUT: Rational maps $\Gamma(s) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in \mathbb{R}$ *and* $\gamma(t) = (x(t),\, y(t))$, $t \in \mathbb{R}$ *parametrizing algebraic curves* $\mathcal{Z} \subset \mathbb{R}^3$ *and* $\mathcal{X} \subset \mathbb{R}^2$.
OUTPUT: The truth of the statement: $\exists [P] \in \mathcal{PP},$ *such that* $\mathcal{X} = P(\mathcal{Z})$.
STEPS:

1. *evaluate* $\Delta_1|_{\mathcal{X}}$ *using (1). If* $\Delta_1|_{\mathcal{X}}(t) \equiv 0$ *follow Algorithm 9 else*
2. *evaluate* $\mathcal{A}(2)$*-invariants (2) on* $\mathcal{X}$. *Obtain two rational functions* $K_{\mathcal{A}}|_{\mathcal{X}}$ *and* $K_{\mathcal{A}}|_{\mathcal{X}}$ *of* t;
3. *define a curve* $\tilde{\mathcal{Z}}$ *parametrized by* $\alpha(s) = (z_2(s), z_3(s))$;
4. *evaluate* $\mathcal{A}(2)$*-invariants (2) on* $\tilde{\mathcal{Z}}$. *Obtain two rational functions* $K_{\mathcal{A}}|_{\tilde{\mathcal{Z}}}$ *and* $T_{\mathcal{A}}|_{\tilde{\mathcal{Z}}}$ *of* s;
5. *determine the truth of the statement:*

$$\forall s \in \mathbb{R} \quad \exists t \in \mathbb{R} \quad K_{\mathcal{A}}|_{\tilde{\mathcal{Z}}}(\,s) = K_{\mathcal{A}}|_{\mathcal{X}}(t) \text{ and } T_{\mathcal{A}}|_{\tilde{\mathcal{Z}}}(\,s) = T_{\mathcal{A}}|_{\mathcal{X}}(t).$$

   If TRUE *exit the procedure, else*

6. *for arbitrary* $b \in \mathbb{R}$, *define a curve* $\tilde{\mathcal{Z}}_b$ *parametrized by* $\beta_b(s) = (z_1(s) + b\, z_2(s),\, z_3(s))$;
7. *evaluate* $\mathcal{A}(2)$*-invariants (2) on* $\tilde{\mathcal{Z}}_b$. *Obtain two rational functions* $K_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_b}(b, s)$ *and* $T_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_b}(b, s)$ *of* b *and* s;
8. *determine the truth of the statement:*

$$\exists b \in \mathbb{R} \quad \forall s \in \mathbb{R} \quad \exists t \in \mathbb{R} \quad K_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_b}(b,\, s) = K_{\mathcal{A}}|_{\mathcal{X}}(t) \text{ and } T_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_b}(b,\, s) = T_{\mathcal{A}}|_{\mathcal{X}}(t).$$

   If TRUE *exit the procedure, else*

9. *for arbitrary* $a \in \mathbb{R}^2$, *define a curve* $\tilde{\mathcal{Z}}_a$ *parametrized by* $\delta_a(s) = (z_1(s) + a_1\, z_3(s),\, z_2 + a_2\, z_3(s))$;
10. *evaluate* $\mathcal{A}(2)$*-invariants (2) on* $\tilde{\mathcal{Z}}_a$. *Obtain two rational functions* $K_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_a}(a, s)$ *and* $T_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_a}(a, s)$ *of* a_1, a_2 *and* s;
11. *determine the truth of the statement:*

$$\exists a \in \mathbb{R}^2 \quad \forall s \in \mathbb{R} \quad t \in \mathbb{R} \quad K_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_a}(a,\, s) = K_{\mathcal{A}}|_{\mathcal{X}}(t) \text{ and } T_{\mathcal{A}}|_{\tilde{\mathcal{Z}}_a}(a,\, s) = T_{\mathcal{A}}|_{\mathcal{X}}(t).$$

Remark 8. *Steps 8 and 11 of Algorithm 7 are computationally challenging real quantifier eliminations problems. Algorithmic solution of such problems implemented, for instance, in* MATHEMATICA *(see command "Reduce"). Our* MAPLE *implementation gives a solution to this problem over the complex numbers. The truth of the statement over the complex numbers provides a necessary but not a sufficient condition for the truth over the real numbers.*

Algorithm 9. (PARALLEL PROJECTIONS TO LINES AND PARABOLAS.)

INPUT: Rational maps $\Gamma(s) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in \mathbb{R}$ *and* $\gamma(t) = (x(t),\, y(t))$, $t \in \mathbb{R}$ *parametrizing algebraic curves* $\mathcal{Z} \subset \mathbb{R}^3$ *and* $\mathcal{X} \subset \mathbb{R}^2$, *such that* $\Delta_1|_{\mathcal{X}}(t) \equiv 0$.

OUTPUT: The truth of the statement: $\exists [P] \in \mathcal{CP},$ *such that* $\mathcal{X} = P(\mathcal{Z})$.

STEPS:

1. *if* $\ddot{x}\,\dot{y} - \ddot{y}\,\dot{x} \equiv 0$ *and* $(\ddot{\Gamma} \times \dot{\Gamma}) \cdot \dddot{\Gamma} \equiv 0$ *return TRUE and exit;*
2. *if* $\ddot{x}\,\dot{y} - \ddot{y}\,\dot{x} \equiv 0$ *and* $(\ddot{\Gamma} \times \dot{\Gamma}) \cdot \dddot{\Gamma} \neq 0$ *return FALSE and exit;*
3. *define a curve* $\tilde{\mathcal{Z}}$ *parametrized by* $\alpha(s) = (z_2(s), z_3(s))$;
4. *evaluate* $\Delta_1|_{\tilde{\mathcal{Z}}}$ *using (1). Obtain a rational function* $\Delta_1|_{\tilde{\mathcal{Z}}}$ *of* s;
5. *determine the truth of the statement:* $\forall s \in \mathbb{R} \quad \Delta_1|_{\tilde{\mathcal{Z}}}(s) = 0$.

   If TRUE *exit the procedure, else*
6. *for arbitrary* $b \in \mathbb{R}$, *define a curve* $\tilde{\mathcal{Z}}_b$ *parametrized by* $\beta_b(s) = (z_1(s) + b\, z_2(s),\, z_3(s))$;
7. *evaluate* $\Delta_1|_{\tilde{\mathcal{Z}}_b}$ *using (1). Obtain a rational function* $\Delta_1|_{\tilde{\mathcal{Z}}_b}$ *of* b *and* s;
8. *determine the truth of the statement:* $\exists b \in \mathbb{R} \quad \forall s \in \mathbb{R} \quad \Delta_1|_{\tilde{\mathcal{Z}}_b}(b, s) = 0$.

   If TRUE *exit the procedure, else*
9. *for arbitrary* $a \in \mathbb{R}^2$, *define a curve* $\tilde{\mathcal{Z}}_a$ *parametrized by* $\delta_a(s) = (z_1(s) + a_1\, z_3(s),\, z_2 + a_2\, z_3(s))$;
10. *evaluate* $\Delta_1|_{\tilde{\mathcal{Z}}_a}$ *using (1). Obtain a rational function* $\Delta_1|_{\tilde{\mathcal{Z}}_a}$ *of* a_1, a_2 *and* s;
11. *determine the truth of the statement:* $\exists a \in \mathbb{R}^2 \quad \forall s \in \mathbb{R} \quad \Delta_1|_{\tilde{\mathcal{Z}}_a}(a, s) = 0$.